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AGNIEL Vidal **Internship : Study of weighted composition operators on Bergman and Hardy spaces**

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INTERNSHIP REPORT

Contents

1 Bergman spaces **2**
1.1 Initial look at Bergman spaces . . . 2

2 Hardy spaces **5**
2.1 Initial look at Hardy Spaces . . . 5
2.2 Reproducing Kernel of a Half-plane and of a Strip . . . 7

3 Reproducing Kernel Hilbert Space theory **8**
3.1 First properties of Reproducing Kernel Hilbert Spaces and Kernel Functions . . . 8
3.2 Interpolation with kernel functions . . . 10

4 Multiplication Operators **12**
4.1 Multiplication operators on $L^2(\Omega)$ . . . 13
4.2 Multiplication operators on $B^2(\Omega)$ and $H^2(\Omega)$ . . . 13

5 Composition Operators **14**
5.1 Composition operators on Hol($\mathbb{D}$) . . . 14
5.2 Bounded composition operators on $H^2(\mathbb{D})$ . . . 16
5.3 Bounded composition operators on $B^2(\mathbb{D})$ . . . 18
5.4 Bounded composition operators on $B^2(\Omega), H^2(\Omega)$ . . . 18

6 Bounded Operators on Hilbert spaces **19**
6.1 Bounded Operators and Polar decomposition theorem . . . 19
6.2 Compact Operators and Hilbert-Schmidt theorem . . . 20

7 Approximation numbers and $\mathcal{I}_p$ Ideals **21**
7.1 Approximation numbers . . . 21
7.2 Operators of type l^p . . . 23
7.3 Operators of type s . . . 26
7.4 Approximation numbers of compact mappings in Hilbert spaces . . . 27
7.5 Trace-class ideal . . . 27
7.6 Hilbert-Schmidt ideal . . . 29

8 Various compact operators in B^2 and H^2 spaces **30**
8.1 Hilbert-Schmidt test for B^2 and H^2 . . . 30
8.2 Compactness criteria in H^2 and B^2 . . . 32
8.3 Relationship between compactness of C_φ and shape of $Im(\varphi)$ . . . 33
8.4 Lens maps and regular polygons in $\mathbb{D}$ . . . 34

9 Composition operators on $H^p(\Omega), B^p(\Omega)$ **35**
9.1 Existence of compact composition operators on $H^p(\Omega), B^p(\Omega)$ . . . 35
9.2 Other properties for composition operators . . . 37

1 Bergman spaces

1.1 Initial look at Bergman spaces

1.1 Definition. — The Bergman space of the Disc, $B^2(\mathbb{D})$, is defined by : $B^2(\mathbb{D}) := Hol(\mathbb{D}) \cap L^2(\mathbb{D})$

1.2 Proposition. — $B^2(\mathbb{D})$ *is an Hilbert space for the inner product :*
$\langle f;g\rangle_{B^2(\mathbb{D})} := \langle f;g\rangle_{L^2(\mathbb{D})} = \iint_{\mathbb{D}} \overline{f(z)}.g(z)|dz|$

Before beginning the demonstration, a certain property coming from holomorphity must be obtained. For f $\in B^2(\mathbb{D})$, for z$\in \mathbb{D}$, $\forall 0 < r < 1-|z|$ the mean value property gives us :

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta}+z)d\theta$$

Thus,

$$\int_0^{1-|z|} f(z)dr = \frac{(1-|z|)^2}{2}f(z) = \frac{1}{2\pi}\int_0^{1-|z|}\int_0^{2\pi} f(re^{i\theta}+z)drd\theta$$

$$\Rightarrow f(z) = \frac{1}{\pi(1-|z|)^2}\iint_{\mathbb{D}(z,1-|z|)} f(x+iy)dxdy$$

$$\Rightarrow |f(z)| \leq \frac{1}{\pi d(z;\mathbb{D}^C)^2}\iint_{\mathbb{D}(z,1-|z|)} |f(x+iy)|dxdy$$

Using Cauchy-Schwarz inequality, we obtain :

$$|f(z)| \leq \frac{1}{\pi d(z;\mathbb{D}^C)^2}\|f\|_{B^2(\mathbb{D}(z,1-|z|))}.\sqrt{\pi}.d(z;\mathbb{D}^C)$$

So we have :

$$|f(z)| \leq \frac{1}{\sqrt{\pi}d(z;\mathbb{D}^C)}\|f\|_{B^2(\mathbb{D})} \tag{1}$$

This equation will be important for the proof of the proposition, but it also shows other properties that $B^2(\mathbb{D})$ possesses.

Proof.
$B^2(\mathbb{D})$ is a vectorial subspace of $L^2(\mathbb{D})$. We will show that $B^2(\mathbb{D})$ is closed for the $\|.\|_{L^2(\mathbb{D})}$ norm.
Let $\{f_n\}_n \in B^2(\mathbb{D})^{\mathbb{N}}$ that converges in $\|.\|_{L^2(\mathbb{D})}$ norm towards $f \in L^2(\mathbb{D})$. Let K be a compact of $\mathbb{D}$.

With (1), we obtain : $\|f_n\|_{L^\infty(K)} \leq \frac{1}{\sqrt{\pi}d(K;\mathbb{D}^C)}\|f_n\|_{L^2(\mathbb{D})}$
Thus, $\{f_n\}_n|_K$ is a Cauchy sequence in $L^\infty(K)$. Since this space is closed, $\{f_n\}_n$ converges uniformly on every compact of $\mathbb{D}$ towards f. Since f_n is holomorphic $\forall n$ and converges uniformly on every compact, its limit is holomorphic. Hence, $f \in Hol(\mathbb{D}) \cap L^2(\mathbb{D}) = B^2(\mathbb{D})$. □

1.3 Proposition. — $\forall z \in \mathbb{D}$, $\begin{array}{rccc}\delta_z : & B^2(\mathbb{D}) & \to & \mathbb{C} \\ & f & \mapsto & f(z)\end{array}$ *is a bounded operator with* $\|\delta_z\| \leq \frac{1}{\sqrt{\pi}d(z;\mathbb{D}^C)}$
Since $B^2(\mathbb{D})$ *is an Hilbert space, Riesz lemma gives a* $k_z \in B^2(\mathbb{D})$ *such as :* $f(z) =< k_z; f >_{B^2(\mathbb{D})}$ $\forall f \in B^2(\mathbb{D})$
We note $K_{B^2(\mathbb{D})}(z;w) = \overline{k_z(w)}$. *Thus,* $f(z) = \iint_{\mathbb{D}} K(z;w)f(w)dxdy$, *for every* $f \in B^2(\mathbb{D})$ $\forall z \in \mathbb{D}$.

$K_{B^2(\mathbb{D})}$ is called the **Reproducing Kernel** of $B^2(\mathbb{D})$. $B^2(\mathbb{D})$ is then called a **Reproducing Kernel Hilbert Space** (RKHS).
$K_{B^2(\mathbb{D})}$ is also called the Bergman Kernel of $\mathbb{D}$.

Before looking a bit further at the Reproducing Kernel, a few other properties of the Bergman spaces need to be seen.

1.4 Proposition. — $\{z^n.\sqrt{\frac{n+1}{\pi}}\}$ *is an orthonormal basis of* $B^2(\mathbb{D})$.

Proof. We have : $\langle z^n; z^m\rangle_{B^2(\mathbb{D})} = \int_0^1\int_0^{2\pi} r^n r^m e^{i\theta(m-n)} r dr d\theta = \left| \begin{array}{l} \frac{2\pi}{n+m+2} = \frac{\pi}{n+1} \text{ if m=n} \\ 0 \text{ else.} \end{array}\right.$

And if we have $\langle z^m; f\rangle_{B^2(\mathbb{D})} = 0, \forall n \geq 0$, then for $f(z) = \sum_{n\geq 0} a_n z^n$ with absolute convergence on $\mathbb{D}$, we obtain

$$\iint\limits_{\mathbb{D}} \overline{z^m} f(z)|dz| = \iint\limits_{\mathbb{D}} \overline{z^m} \sum_{n\geq 0} a_n z^n |dz| = \sum_{n\geq 0} a_n \iint\limits_{\mathbb{D}} \overline{z^m} z^n |dz| = a_n \frac{\pi}{n+1} = 0, \forall n \geq 0.$$

Thus, $f \equiv 0$. □

The arguments used to show that $B^2(\mathbb{D})$ is an Hilbert space with a Reproducing Kernel can also be used for Bergman spaces of other domains.

1.5 Proposition. — *Let* Ω *be an open and simply connected region of* $\mathbb{C}$, $\Omega \neq \mathbb{C}$.
Then, $B^2(\Omega)$ *is an Hilbert space with a Reproducing Kernel.*

1.6 Theorem. *Riemann Mapping Theorem — If* Ω *is an open and simply connected region of* $\mathbb{C}$ *that isn't* $\mathbb{C}$*, then there exists* $\varphi : \Omega \to \mathbb{D}$ *a biholomorphism.*
This biholomorphism is uniquely determined if we fix the image of two points.

With this theorem, for $f, g \in B^2(\Omega)$, we have :

$$\langle f; g\rangle_{B^2(\Omega)} = \iint\limits_{\Omega} \overline{f(z)} g(z) dx dy = \iint\limits_{\mathbb{D}} \overline{f(\varphi^{-1}(w))} g(\varphi^{-1}(w)) |(\varphi^{-1})'(w)|^2 dx' dy'$$

1.7 Proposition. — *For* $\varphi : \Omega \to \mathbb{D}$ *a biholmorphism,* $\begin{array}{llll} U: & B^2(\Omega) & \to & B^2(\mathbb{D}) \\ & f & \mapsto & (f\circ\varphi^{-1}).(\varphi^{-1})' \end{array}$
is an unitary map : $\langle Uf; Ug\rangle_{B^2(\mathbb{D})} = \langle f; g\rangle_{B^2(\Omega)}$,
and $U^{-1}h = (h \circ \varphi).\varphi'$.

1.8 Note. $B^2(\mathbb{C}) = \{0\}$*, as an holomorphic function in* $B^2(\mathbb{C})$ *verifies :*

$$f^2(z) = \frac{2}{1-R^2}.\frac{1}{2\pi}\int_0^R\int_0^{2\pi} f^2(r.e^{i\theta}+z) r.dr d\theta,\ \forall R > 0,\ \forall z \in \mathbb{C}$$

$$\Rightarrow |f(z)|^2 \leq \frac{1}{\pi.(1-R^2)}.\iint\limits_{\mathbb{C}} |f(x+iy)|^2 dx dy\ ,\ \forall R > 0,\ \forall z \in \mathbb{C}$$

$$\Rightarrow f(z) = 0\ ,\ \forall z \in \mathbb{C}$$

1.9 Note. *With an unitary U between* $B^2(\Omega)$ *and* $B^2(\mathbb{D})$, $\{U^{-1}(z^n.\sqrt{\frac{n+1}{\pi}})\}_n$ *is an orthonormal basis of* $B^2(\Omega)$.

1.10 Proposition. — *The Reproducing Kernel $K_{B^2(\Omega)}$ is unique.*
If there exists $\widetilde{K} : \Omega \times \Omega \to \mathbb{C}$ such as :
$i)\overline{\widetilde{K}(z;.)} \in B^2(\Omega)$
$ii) f(z) = \langle \overline{\widetilde{K}(z;.)}; f\rangle_{B^2(\Omega)}$, $\forall z \in \Omega, \forall f \in B^2(\Omega)$
then $\tilde{K} \equiv K_{B^2(\Omega)}$.

Proof. We have : $\overline{\widetilde{K}(z;w)} = \langle \overline{K_{B^2(\Omega)}(w;.)}; \overline{\widetilde{K}(z;.)}\rangle = \overline{\langle \overline{\widetilde{K}(z;.)}; \overline{K_{B^2(\Omega)}(w;.)}\rangle} = K_{B^2(\Omega)}(w;z)$. And $K_{B^2(\Omega)}(w;z) = \overline{\langle \overline{K_{B^2(\Omega)}(z;.)}; \overline{K_{B^2(\Omega)}(w;.)}\rangle} = \langle \overline{K_{B^2(\Omega)}(w;.)}; \overline{K_{B^2(\Omega)}(z;.)}\rangle = \overline{K_{B^2(\Omega)}(z;w)}$. □

1.11 Note. *We also saw that $K_{B^2(\Omega)}(z;w) = \overline{K_{B^2(\Omega)}(w;z)}$.*
Thus, $K_{B^2(\Omega)}(z;w)$ is holomorphic in z and antiholomorphic in w.

1.12 Proposition. — *For $\{\psi_n\}_n$ an orthonormal basis of $B^2(\Omega)$, we have :*

$$K_{B^2(\Omega)}(z;w) = \sum_{n\geq 0} \overline{\psi_n(w)}.\psi_n(z)$$

Proof. $\overline{K_{B^2(\Omega)}(z;.)}$ is in $B^2(\Omega)$. Thus, $\overline{K_{B^2(\Omega)}(z;w)} = \sum_{n\geq 0} \psi_n(w).b_{n,z}$.
And $b_{n,z} = \langle \psi_n; \overline{K_{B^2(\Omega)}(z;.)}\rangle = \overline{\langle \overline{K_{B^2(\Omega)}(z;.)}; \psi_n\rangle} = \overline{\psi_n(z)}$ □

1.13 Note. *On a Reproducing Kernel Hilbert Space, the series $\sum_{n\geq 0} \overline{\psi_n(w)}.\psi_n(z)$ is independant of the orthonormal basis chosen.*
Thus, knowing explicitly an orthonormal basis is enough to determine the reproducing kernel.
Also, $K_{B^2(\Omega)}(z;z)$ is always real positive.

1.14 Proposition. — *We can now calculate the Bergman Kernel of the disc :*

$$K_{B^2(\mathbb{D})}(z;w) = \sum_{n\geq 0} (z\overline{w})^n.\frac{n+1}{\pi} = \frac{1}{\pi}.\frac{1}{(1-\overline{w}z)^2} \tag{2}$$

1.15 Proposition. — *For Ω_1, Ω_2 open simply connected regions of $\mathbb{C}$, for $\varphi : \Omega_1 \to \Omega_2$ a biholomorphism, and $U : B^2(\Omega_1) \to B^2(\Omega_2)$ the associated unitary, we have :*

$$K_{B^2(\Omega_1)}(z;w) = \overline{k_z^{\Omega_1}(w)} = \overline{(U^{-1}k_{\varphi(z)}^{\Omega_2})(w).\varphi'(z)} = \varphi'(z).K_{B^2(\Omega_2)}(\varphi(z);\varphi(w)).\overline{\varphi'(w)}$$

1.16 Note. *The Bergman kernel of the disc has no zeroes.*
$K_{B^2(\mathbb{D})}(z;z)$ has a minimum of $\frac{1}{\pi}$ and grows towards infitiny when z goes towards $\partial\mathbb{D}$.
For $\varphi : \Omega \to \mathbb{D}$ a biholomorphism, φ' has no zeroes on Ω, so the Bergman kernel of Ω has no zeroes too.
However, $K_{B^2(\Omega)}(z;z)$ may not grow towards infinity for $z \to \partial\Omega$, depending on the behaviour of $|\varphi'(z)|^2$, which depends on the shape of Ω.
We also have :

$$K_{B^2(\Omega)}(z;\varphi^{-1}(0)) = \frac{1}{\pi}.\frac{\varphi'(z).\overline{\varphi'(\varphi^{-1}(0))}}{(1-\overline{\varphi(\varphi^{-1}(0))}.z)^2} = \frac{\varphi'(z).\overline{\varphi'(\varphi^{-1}(0))}}{\pi}$$

1.17 Theorem. *Bergman Projection* —
Let $B : L^2(\Omega) \to B^2(\Omega)$ be the orthogonal projection over $B^2(\Omega)$, and let $k_z^\Omega(.) = \overline{K_{B^2(\Omega)}(z;.)} \in B^2(\Omega)$.
Thus, $\forall f \in L^2(\Omega)$, $\forall z \in \Omega$, $Bf \in B^2(\Omega)$ and :

$$Bf(z) = \langle k_z^\Omega; Bf\rangle_{B^2(\Omega)} = \langle B^*k_z^\Omega; f\rangle_{L^2(\Omega)} = \langle Bk_z^\Omega; f\rangle_{L^2(\Omega)}$$

$$\Rightarrow Bf(z) = \langle k_z^\Omega; f\rangle_{L^2(\Omega)}$$

See [7], Classical spaces of holomorphic functions, Ch 1-2.

1.18 Note. *The Bergman space $B^p(\Omega)$, for $0 < p < \infty$ can also be defined similarly to $B^2(\Omega)$. For $p \neq 2$, B^p is only a Banach space, but shares certain properties with B^2.*
However, this internship focuses on operators on certain Hilbert spaces, so the B^p theory won't be developed.

2 Hardy spaces

2.1 Initial look at Hardy Spaces

2.1 Definition. — The Hardy space of the disc, $H^2(\mathbb{D})$, is defined by :

$$H^2(\mathbb{D}) := \{f \in Hol(\mathbb{D}) \text{ such as } f(z) = \sum_{n\geq 0} a_n.z^n \text{ with } \sum_{n\geq 0} |a_n|^2 < \infty\}$$

For $f, g \in H^2(\mathbb{D})$, $f(z) = \sum_{n\geq 0} a_n.z^n$, $g(z) = \sum_{n\geq 0} b_n.z^n$, we define :

$$\langle f; g\rangle_{H^2(\mathbb{D})} := \sum_{n\geq 0} \overline{a_n}.b_n$$

And $\|f\|_{H^2(\mathbb{D})} := \sqrt{\sum_{n\geq 0} |a_n|^2}$.

With these definitions, $H^2(\mathbb{D})$ is a normed vectorial space.
We notice that the map : $\begin{array}{cccc} T: & l^2 & \to & H^2(\mathbb{D}) \\ & \{a_n\}_n & \mapsto & (z \mapsto \sum_{n\geq 0} a_n.z^n) \end{array}$ is a bijective isometry from l^2 to $H^2(\mathbb{D})$.
This gives us the following result :

2.2 Theorem. — *$(H^2(\mathbb{D}), \langle .; .\rangle_{H^2(\mathbb{D})})$ is an Hilbert space.*

2.3 Proposition. — *For $f \in H^2(\mathbb{D})$, $f(z) = \sum_{n\geq 0} a_n.z^n$, the function : $\begin{array}{cccc} \widetilde{f}: & \partial\mathbb{D} & \to & \widehat{\mathbb{C}} \\ & z & \mapsto & \sum_{n\geq 0} a_n.z^n \end{array}$ belongs to $L^2(\partial\mathbb{D})$.*

The next theorem gives other expressions of the inner product of $H^2(\mathbb{D})$ that will be very useful to show that the Hardy space shares similar properties with the Bergman space.

2.4 Theorem. — *For $f, g \in H^2(\mathbb{D})$, we have :*

$$\langle f; g\rangle_{H^2(\mathbb{D})} = \lim_{r\to 1^-} (\frac{1}{2\pi}.\int_0^{2\pi} \overline{f(r.e^{i\theta})}.g(r.e^{i\theta})d\theta) = \langle \widetilde{f}; \widetilde{g}\rangle_{L^2(\partial\mathbb{D})} \tag{3}$$

With this, we can now define the Hardy space of an open space Ω, and prove that these Hardy spaces are Reproducing Kernel Hilbert Spaces.

2.5 Definition. — Let Ω be an open simply connected space, $\Omega \neq \mathbb{C}$. We have a biholomorphism $\varphi : \mathbb{D} \to \Omega$.
For $0 < r < 1$ we define : $\varphi(\gamma_r) : t \in [0; 2\pi] \mapsto \varphi(r.e^{i.t}) \in \Omega$.
Thus, for $g : \Omega \to \mathbb{C}$, we have : $\int_{\varphi(\gamma_r)} g(z)|dz| := \int_0^{2\pi} g(\varphi(r.e^{i\theta})).|\varphi'(r.e^{i\theta})|d\theta$
The Hardy space of Ω, $H^2(\Omega)$, is defined by :

$$H^2(\Omega) := \{f \in Hol(\Omega) \text{ such as } \lim_{r\to 1^-} (\int_{\varphi(\gamma_r)} |f(z)|^2|dz|) < \infty\}$$

At first, $H^2(\Omega)$ is a vectorial space.
Since $\varphi : \mathbb{D} \to \Omega$ is a biholomorphism, $\varphi' : \mathbb{D} \to \mathbb{C}$ is holomorphic and has no zeroes. Thus, since $\mathbb{D}$ is simply connected, there exists $\phi \in Hol(\mathbb{D})$ such as $\varphi' = exp \circ \phi$. Thus, $\Phi : z \in \mathbb{D} \mapsto exp(\frac{\phi(z)}{2})$ is holomorphic and verifies $\Phi(z)^2 = \varphi'(z) \Rightarrow |\varphi'| = \overline{\Phi}.\Phi$
Thus, the map $V : f \in H^2(\Omega) \mapsto (f \circ \varphi).\Phi \in H^2(\mathbb{D})$ is well defined and bijective, thanks to theorem 2.4. It allows us to define an inner product $\langle ; \rangle_{H^2(\Omega)}$ from $\langle ; \rangle_{H^2(\mathbb{D})}$ that makes $H^2(\Omega)$ an Hilbert space for which V is an unitary bijective map.
The inverse of V is $V^{-1} : f \in H^2(\mathbb{D}) \mapsto (f \circ \varphi^{-1}).\widetilde{\Phi} \in H^2(\Omega)$, with $\widetilde{\Phi} \in Hol(\Omega)$ such as $\widetilde{\Phi}^2 = (\varphi^{-1})'$.

For $f, g \in H^2(\Omega)$, we have :

$$\langle f; g \rangle_{H^2(\Omega)} := \langle Vf; Vg \rangle_{H^2(\mathbb{D})} = \lim_{r \to 1^-} (\frac{1}{2\pi} \int_{\varphi(\gamma_r)} \overline{f(z)}.g(z)|dz|)$$

$$\|f\|_{H^2(\Omega)} := \|Vf\|_{H^2(\mathbb{D})} = \sqrt{\frac{1}{2\pi} \lim_{r \to 1^-} (\int_0^{2\pi} |f(\varphi(r.e^{i\theta}))|^2.|\Psi(r.e^{i\theta})|^2 d\theta)} = \sqrt{\frac{1}{2\pi} \lim_{r \to 1^-} (\int_{\varphi(\gamma_r)} |f(z)|^2 |dz|)}$$

$$\|f\|^2_{H^2(\Omega)} = \frac{1}{2\pi} \lim_{r \to 1^-} (\int_0^{2\pi} |f(\varphi(r.e^{i\theta}))|^2.|\Psi(r.e^{i\theta})|^2 d\theta) = \frac{1}{2\pi} \sup_{0<r<1} (\int_0^{2\pi} |f(\varphi(r.e^{i\theta}))|^2.|\Psi(r.e^{i\theta})|^2 d\theta)$$

As of now, the definition of the elements of $H^2(\Omega)$ depends on the choice of the biholomorphism φ. We will show later that the set $H^2(\Omega)$ doesn't depend on the choice of φ. The inner product on $H^2(\Omega)$ does depend on the choice of φ, but it will be shown that they are all equivalent.

We will now show that $H^2(\mathbb{D})$ is a RKHS.

2.6 Proposition. — *$\forall z \in \mathbb{D}$, $\begin{array}{rccc} \delta_z : & H^2(\mathbb{D}) & \to & \mathbb{C} \\ & f & \mapsto & f(z) \end{array}$ is a bounded operator with $\|\delta_z\| \leq \frac{1}{\sqrt{\pi} d(z;\mathbb{D}^C)}$*
Thus, $H^2(\mathbb{D})$ possesses a Reproducing Kernel $K_{H^2(\mathbb{D})}$, with $f(z) = \langle \overline{K_{H^2(\mathbb{D})}(z; .)}; f \rangle_{H^2(\mathbb{D})}$, $\forall f \in H^2(\mathbb{D})$, $\forall z \in \mathbb{D}$.

Proof. Let $f \in H^2(\mathbb{D})$, $z \in \mathbb{D}$. Using Cauchy's formula, we have :

$$f(z) = \frac{1}{2i\pi} \int_{\gamma_r} \frac{f(w)}{w - z} dw = \frac{1}{2\pi} . \int_0^{2\pi} \frac{f(r.e^{i\theta})}{r.e^{i\theta} - z} r.e^{i\theta}.d\theta \text{ for } |z| < r < 1 \,,$$

$$\Rightarrow |f(z)| \leq \sqrt{\frac{1}{2\pi} . \int_0^{2\pi} |f(r.e^{i\theta})|^2 d\theta} . \sqrt{\frac{1}{2\pi} . \int_0^{2\pi} \frac{r^2}{|r.e^{i\theta} - z|^2} d\theta} \leq \|f\|_{H^2(\mathbb{D})} . \frac{r}{r - |z|}$$

$$\Rightarrow |f(z)| \leq \|f\|_{H^2(\mathbb{D})} . \frac{1}{1 - |z|} \tag{4}$$

□

2.7 Note. *A similar proof can be done for $H^2(\Omega)$ by considering integrals on $\varphi(\gamma_r)$ for $0 < r - |z| < d(z; \Omega^C)$ instead of integrals on γ_r for $0 < r - |z| < 1 - |z|$*

For $H^2(\mathbb{C}) := \{f \in Hol(\mathbb{C})$ such as $\sup_{0<r<\infty} (\int_{\gamma_r} |f(z)|^2 |dz|) < \infty\}$, we have $H^2(\mathbb{C}) = \{0\}$.
An f in $H^2(\mathbb{C})$ would verify $|f(z)| \leq \|f\|_{H^2(\mathbb{C})} . \frac{r}{r-|z|}$ $\forall 0 < r - |z| \Rightarrow |f| \leq 2.\|f\|_{H^2(\mathbb{C})} < \infty \Rightarrow f$ is bounded on $\mathbb{C} \Rightarrow f$ is constant on $\mathbb{C} \Rightarrow f \equiv 0$.

With the same arguments used for the Bergman Kernel, we now obtain :

2.8 Proposition. —
For $\{\psi_n\}_n$ an orthonormal basis of $H^2(\Omega)$, we have :

$$K_{B^2(\Omega)}(z;w) = \sum_{n\geq 0} \overline{\psi_n(w)}.\psi_n(z)$$

In the case of $H^2(\mathbb{D})$, we have :

$$K_{B^2(\mathbb{D})}(z;w) = \sum_{n\geq 0} (\overline{w}.z)^n = \frac{1}{1-\overline{w}.z} \tag{5}$$

*$K_{B^2(\mathbb{D})}$ is called the **Cauchy-Szegö** kernel.*

2.9 Proposition. — *For Ω an open simply connected space, $\varphi : \Omega \to \mathbb{D}$ a biholomorphism, $\Psi \in Hol(\mathbb{D})$ such as $\Psi^2 = \varphi'$, the reproducing kernel of $H^2(\Omega)$ verifies :*

$$K_{H^2(\Omega)}(z;w) = \Psi(z).K_{H^2(\mathbb{D})}(\varphi(z);\varphi(w)).\overline{\Psi(w)}$$

2.10 Proposition. — *Since $\frac{1}{\pi}.K_{H^2(\mathbb{D})}(z;w)^2 = K_{B^2(\mathbb{D})}(z;w)$ and $\Psi(z)^2 = \varphi'(z)$, for any Ω open and simply connected we have :*

$$\frac{1}{\pi}.K_{H^2(\Omega)}(z;w)^2 = K_{B^2(\Omega)}(z;w)$$

2.11 Proposition. — *$H^2(\mathbb{D}) \subset B^2(\mathbb{D})$, but $H^2(\mathbb{D}) \neq B^2(\mathbb{D})$*

Proof.
We have $H^2(\mathbb{D}) = \{f \in Hol(\mathbb{D})$ such as $f(z) = \sum_{n\geq 0} a_n.z^n$ with $\{a_n\}_n \in l^2\}$ and $B^2(\mathbb{D}) = \{f \in Hol(\mathbb{D})$ such as $f(z) = \sum_{n\geq 0} b_n.z^n.\sqrt{\frac{n+1}{\pi}}$ with $\{b_n\}_n \in l^2\}$
For $\{a_n\}_n \in l^2$, $\{a_n.\sqrt{\frac{\pi}{n+1}}\} \in l^2$.
But for $b_n = \frac{1}{n+1}$, $\{b_n\}_n \in l^2$ but $\{b_n.\sqrt{\frac{n+1}{\pi}}\}_n \notin l^2$, so for $f(z) = \sum_{n\geq 0} b_n.z^n.\sqrt{\frac{n+1}{\pi}}$, $f \in B^2(\mathbb{D})$ but $f \notin H^2(\mathbb{D})$.

□

2.12 Note. *The Bergman space $H^p(\Omega)$[1], for $0 < p < \infty$ can also be defined similarly to $H^2(\Omega)$. For $p \neq 2$, H^p is only a Banach space, but shares certain properties with H^2.*
However, this internship focuses on operators on certain Hilbert spaces, so the H^p theory won't be developed.

2.2 Reproducing Kernel of a Half-plane and of a Strip

We define $\Omega_1 := \{z \in \mathbb{C}$ such as $Re(z) > 0\}$ a half-plane, and $\Omega_2 := \{z \in \mathbb{C}$ such as $-\frac{\pi}{2} < Im(z) < \frac{\pi}{2}\}$ a strip of width π.
$\psi_1 : z \in \Omega_1 \mapsto \frac{1-z}{1+z} \in \mathbb{D}$ is a biholomorphism as it is an homography that sends $i\mathbb{R}$ to $\partial\mathbb{D}$ and that sends 1 to 0.
$\psi_2 : z \in \Omega_2 \mapsto exp(z) \in \Omega_1$ is a biholomorphism too.
We have $\psi_1'(z) = \frac{2}{(1+z)^2}$, $\psi_2' = exp(z)$, and $\psi_3 := \psi_1 \circ \psi_2(z) = \frac{1-e^z}{1+e^z} = \tanh(\frac{z}{2})$ with $\psi_3'(z) = \frac{1}{2\cosh(\frac{z}{2})^2}$

[1]See [6], Theory of H^p Spaces.

With this, we can now compute the reproducing kernels for these two spaces :

$$K_{H^2(\Omega_1)}(z,w) = (\psi_1')^{\frac{1}{2}}(z).K_{H^2(\mathbb{D})}(\psi_1(z);\psi_1(w)).\overline{(\psi_1')}^{\frac{1}{2}}(w) = \frac{\sqrt{2}}{(1+z)}\frac{\sqrt{2}}{\overline{(1+w)}}\frac{1}{1-\frac{\overline{1-w}}{1+w}\frac{1-z}{1+z}}$$
$$= \frac{2}{\overline{(1+w)}(1+z) - \overline{(1-w)}(1-z)} = \frac{2}{2(\overline{w}+z)} = \frac{1}{\overline{w}+z}$$

With Proposition 2.10 we obtain : $K_{B^2(\Omega_1)}(z,w) = \frac{1}{\pi}\frac{1}{(\overline{w}+z)^2}$
We also have : $K_{H^2(\Omega_1)}(z,z) = \frac{1}{2Re(z)}$

$$K_{B^2(\Omega_2)}(z,w) = \frac{1}{\pi}\frac{1}{2\overline{\cosh(\frac{w}{2})}}\frac{1}{2\cosh(\frac{z}{2})}\frac{1}{(1-\overline{\tanh(\frac{w}{2})}\tanh(\frac{z}{2}))^2} = \frac{1}{4\pi}\frac{1}{(\overline{\cosh(\frac{w}{2})}\cosh(\frac{z}{2}) - \overline{\sinh(\frac{w}{2})}\sinh(\frac{z}{2}))^2}$$
$$= \frac{1}{4\pi}\frac{1}{(\cosh(\frac{\overline{w}}{2})\cosh(\frac{z}{2}) - \sinh(\frac{\overline{w}}{2})\sinh(\frac{z}{2}))^2} = \frac{1}{4\pi}\frac{1}{(\cosh(\frac{\overline{w}-z}{2}))^2}$$

With Proposition 2.10 we obtain : $K_{H^2(\Omega_2)}(z,w) = \frac{1}{2}\frac{1}{\cosh(\frac{\overline{w}-z}{2})}$
We also have : $K_{H^2(\Omega_2)}(z,z) = \frac{1}{2}\frac{1}{\cosh(iIm(z))} = \frac{1}{2\cos(Im(z))}$

- Now that a primary description of the Hardy and Bergman spaces has been made, we will focus on the RKHS theory before looking at certain operators on B^2 and H^2.

3 Reproducing Kernel Hilbert Space theory

3.1 First properties of Reproducing Kernel Hilbert Spaces and Kernel Functions

3.1 Definition. Let X be a set, and let H be a Hilbert space of functions from X to $\mathbb{C}$.
If $\begin{array}{rccc}\delta_x: & H & \to & \mathbb{C} \\ & f & \mapsto & f(x)\end{array}$ is bounded for all $x \in X$, then the Riesz Lemma applied to δ_x gives us k_x^H in H such as $\langle k_x^H;.\rangle_H = \delta_x(.)$.
This gives us the **Reproducing Kernel** of H, $\begin{array}{rccc}K_H: & X\times X & \to & \mathbb{C} \\ & (x;y) & \mapsto & \overline{k_x^H(y)}\end{array}$, who verifies :
i) $\overline{K_H(x;.)} \in H$, $\forall x \in X$
ii) $\forall f \in H$, $\forall x \in X$, $f(x) = \langle \overline{K_H(x;.)}; f\rangle_H$
The Hilbert space H is then called a **Reproducing Kernel Hilbert Space (RKHS)**.

3.2 Proposition. — *Let H be a RKHS with a reproducing kernel K_H. Then, we have :*
- $K_H(x;y) = \overline{K_H(y;x)}$, $\forall x,y \in X$
- K_H is unique : if there is $\widetilde{K} : X\times X \to \mathbb{C}$ that verifies i) and ii), then $\widetilde{K} = K_H$.
- $\forall x,y \in X$, $K_H(x;y) = \overline{k_x^H(y)} = \overline{\langle k_y^H; k_x^H\rangle_H} = \langle k_x^H; k_y^H\rangle_H$.
- Let $\{\psi_n\}_n$ be an orthonormal basis of H. Then $K_H(x;y) = \sum_{n\geq 0}\psi_n(x).\overline{\psi_n(y)}$
This series doesn't depend on the orthonormal basis chosen.
- $K_H(x;x) = \sum_{n\geq 0}|\psi_n(x)|^2 \geq 0$, and $\sqrt{K_H(x;x)} = \|k_x^H\|_H = \|\delta_x\|$.
- $\forall N \geq 0$, $\forall a_0,\dots,a_N \in \mathbb{C}$, $\forall z_0,\dots,z_N \in X$,

$$\sum_{n,m=0}^{N}\overline{a_n}.a_m.K_H(z_n;z_m) = \sum_{n,m=0}^{N}\overline{a_n}.a_m.\overline{\langle k_{z_n}^H; k_{z_m}^H\rangle_H} = \overline{\langle \sum_{n=0}^{N} a_n.k_{z_n}^H; \sum_{n=0}^{N} a_n.k_{z_n}^H\rangle_H} = \|\sum_{n=0}^{N} a_n.k_{z_n}^H\|_H^2 \geq 0$$

- If H' is a closed subspace of H, then H' is also a RKHS.

$\forall x \in X$, $k_x^{H'} = Pk_x^H$, with P the orthogonal projection on H'.
- $\{k_x^H\}_x$ is dense in H. If $\forall x \in X$, $\langle k_x^H; f\rangle_H = 0$, then $f(x) = 0 \; \forall x \in X \Rightarrow f = 0$.

Proof. Several points have already been proved in the case of a Bergman or Hardy space, and the proof for a general RKHS H is the same.
Two points are left to prove :
- $\|\delta_x\| = sup_{\|f\|_H=1}|f(x)| = sup_{\|f\|_H=1}|\langle k_x^H; f\rangle_H| \leq 1.\|k_x^H\|_H$, and $\delta_x(k_x^H) = \langle k_x^H; k_x^H\rangle_H = \|k_x^H\|_H^2$.
- Let H' be a closed subspace of H, and let P be the orthogonal projection on H'.
Then, $\forall f \in H'$,$\forall x \in X$, $Pk_x^H \in H'$ and $\langle Pk_x^H; f\rangle_H = \langle k_x^H; Pf\rangle_H = \langle k_x^H; f\rangle_H = f(x)$.

□

3.3 Note. *Knowing that $\{z^n.\frac{n+1}{\pi}\}_n$ is an orthonormal basis of $B^2(\mathbb{D})$, and that $\{z^n\}_n$ is an orthonormal basis of $H^2(\mathbb{D})$, we obtained $K_{B^2(\mathbb{D})(z;w)} = \frac{1}{\pi}\frac{1}{(1-z.\overline{w})^2}$ and $K_{H^2(\mathbb{D})(z;w)} = \frac{1}{(1-z.\overline{w})}$.*
We now know that for $z \in \mathbb{D}$, $\|\delta_z^{B^2(\mathbb{D})}\| = \frac{1}{\pi}\frac{1}{(1-|z|^2)^2}$ and $\|\delta_z^{H^2(\mathbb{D})}\| = \frac{1}{(1-|z|^2)}$.

We will now try to look at the concept in the converse way :
Given a set X, and a function $K : X \times X \to \mathbb{C}$ that has a specific property, does there exist an Hilbert space H of functions from X to $\mathbb{C}$ such as K is the reproducing kernel of H ?

3.4 Definition. Let X be a set. A function $K : X \times X \to \mathbb{C}$ that verifies :
i) $\forall N \geq 0, \forall a_0, \ldots, a_N \in \mathbb{C}, \forall z_0, \ldots, z_N \in X, \sum_{n,m=0}^N \overline{a_n}.a_m.K(z_n; z_m) \geq 0$
is called a **Kernel function** on X.

3.5 Note. *The kernel function property is equivalent to : $\forall n \geq 0$, $\forall x_0, \ldots, x_n \in X$, $(K(x_i, x_j))_{i,j}$ is self-adjoint positive.*
In this case, we say that $(K(x, y))$ is self-adjoint positive.
In the case of the Bergman and Hardy spaces, we even saw that their reproducing kernels had no zeroes, which means that the matrixes $(K(x_i, x_j))_{i,j}$ are all self-adjoint definite positive.
However, this is not always true.

3.6 Theorem. *Let X be a set. Let K be a kernel function on X.*
Then, there exists an unique Hilbert space H(K) of functions from X to $\mathbb{C}$ for which K is its reproducing kernel.

We will first see two lemmas about RKHS to help proving the fact that the Hilbert space that will be built is an Hilbert space of functions from X to $\mathbb{C}$, and that it is unique.

3.7 Lemma. *Let H be a RKHS on X. Let $\{f_n\}_n \in H^{\mathbb{N}}$. If $f_n \to_{n\to\infty}^{\|.\|_H} f$, then $f_n(x) \to_{n\to\infty} f(x) \; \forall x \in X$, as δ_x is a bounded operator $\forall x \in X$.*

3.8 Lemma. *Let H_1, H_2 be RKHS on X, with $K_{H_1} = K_{H_2}$.*
Then $H_1 = H_2$ and $\|.\|_{H_1} = \|.\|_{H_2}$.

Proof. Since $K_{H_1} = K_{H_2} = K$, we have $k_x^{H_1} = k_x^{H_2} = k_x$, $\forall x \in X$. As $\{k_x\}_x$ is dense in H_1 and H_2, for $f \in H_i$, we have a countable sequence $(\alpha_y)_y$ such as $f(x) = \sum_{y\in X} \alpha_y.k_y(x)$.
Thus, $\|f\|_{H_i}^2 = \sum_y \sum_z \overline{\alpha_y}.\alpha_z.\langle k_y; k_z\rangle_{H_i} = \sum_y \sum_z \overline{\alpha_y}.\alpha_z.K(y; z)$.
So $H_1 = \overline{Vect(\{k_x\}_x)}^{\|.\|_{H_1}} = \overline{Vect(\{k_x\}_x)}^{\|.\|_{H_2}} = H_2$, and $\|.\|_{H_1} = \|.\|_{H_2}$.

□

See [5], An introduction to the theory of Reproducing Kernel Hilbert Spaces, Ch 1-5.

Proof. Theorem
For K a kernel function, for $x, y \in X$, we note $k_x(y) = \overline{K(x;y)}$. Thus, $V = Vect(\{k_x\}_x)$ is a vectorial space of functions from X to $\mathbb{C}$.
$\langle k_x; k_y \rangle = K(y;x)$ is an inner product on V, thanks to the property of the Kernel function. Even if $(K(x,y))$ is only self-adjoint positive, we will obtain definiteness on $\langle ; \rangle$ thanks to the diagonalization of this matrix on a finite set $\{x_1, \ldots, x_n\}$.
Thus, $H(K) = \overline{V}^{\langle .;. \rangle}$ is an Hilbert space and a space of functions from X to $\mathbb{C}$, because convergence for $\|.\|$ on V implies ponctual convergence.
Thus, H(K) is an RKHS for which K is a reproducing kernel. The second lemma gives the uniqueness of such a H(K). □

3.9 Example. *Let $f : X \to \mathbb{C}$ a function. For $K(x,y) = f(x).\overline{f(y)}$, $(K(x,y))$ is self-adjoint positive, $H(K) = Vect(f)$, and if $f \neq 0$ then $\|f\|_{H(K)} = 1$.*
This exemple also shows that any set X possesses kernel functions.

- For $(\alpha_i)_i \in \mathbb{C}$, $(x_i)_i \in X$ we have $\sum_{i,j} \overline{\alpha_i}.\alpha_j.K(x_i, x_j) = |\sum_i \alpha_i.f(x_i)|^2 \geq 0$, so $(K(x,y))$ is indeed self-adjoint positive. With theorem 3.6, we have a RKHS H(K) whose reproducing kernel is K.
- $\forall y \in X$, $k_y(.) = \overline{f(y)}.f(.)$, so $k_y \in Vect(f) \Rightarrow H(K) = Vect(f)$, as $Vect(\{k_y\}_y) = Vect(f)$ is dense in $H(K)$.
- Let $y \in X$ such as $f(y) \neq 0$. Then $|f(y)|^2.\|f\|^2_{H(K)} = \|\overline{f(y)}.f\|^2_{H(K)} = \|k_y\|^2_{H(K)} = K(y,y) = |f(y)|^2$. So $\|f\|_{H(K)} = 1$.

3.2 Interpolation with kernel functions

Let X be a set, and let K be a kernel function on that set. Let $H(K)$ be the RKHS associated to K.
Let $F = \{x_1, \ldots, x_n\} \subset X$ a finite subset. We define $H_F := Vect(\{k_{x_1}, \ldots, k_{x_n}\})$, and P_F the orthogonal projection onto H_F.
We want to look at certain properties of that subspace H_F, and use elements of this space to interpolate or approximate functions in $H(K)$.

3.10 Note. —
- $dim(H_F) \leq n$.
- $dim(H_F) < n \Leftrightarrow \exists(\alpha_1, \ldots, \alpha_n) \neq (0, \ldots, 0)$ such as $\sum \alpha_i.k_{x_i} = 0$.
$\Leftrightarrow \exists(\alpha_1, \ldots, \alpha_n) \neq (0, \ldots, 0)$ such as $\forall f \in H(K)$, $0 = \langle \sum \alpha_i.k_{x_i}; f \rangle = \sum \overline{\alpha_i}.f(x_i)$
- $g \in H_F^{\perp} \Leftrightarrow g(x_i) = \langle k_{x_i}; g \rangle = 0$, $\forall 1 \leq i \leq n$.
- Thus, $P_F(g)(x_i) = g(x_i)$, $\forall 1 \leq i \leq n$, $\forall g \in H(K)$.

3.11 Proposition. — *Let $F = \{x_1, \ldots, x_n\} \subset X$, $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$. If there exists $g \in H(K)$ that interpolates $\{\lambda_i\}_i$ on $\{x_i\}_i$, then $P_F(g)$ is the function of minimal norm in $H(K)$ that does the interpolation.*

Proof. For g_1 that also interpolates the values, we have $(g_1 - P_F(g)) \in H_F^{\perp}$.
Thus, we obtain $\|g_1\|^2 = \|g_1 - P_F(g)\|^2 + \|P_F(g)\|^2 \geq \|P_F(g)\|^2$. □

3.12 Proposition. — *Let $F = \{x_1, \ldots, x_n\} \subset X$.*
If $(\alpha_1, \ldots, \alpha_n) \in Ker((K(x_i, x_j))_{i,j})$ then $\sum \alpha_i.k_{x_i} = 0$

Proof. For $f = \sum \alpha_i.k_{x_i}$, we have :
$\|f\|^2 = \sum_{i,j} \overline{\alpha_i}.\alpha_j.\langle k_{x_i}; k_{x_j} \rangle = \sum_{i,j} \overline{\alpha_i}.\alpha_j.K(x_i; x_j) = \overline{(\alpha_1, \ldots, \alpha_n)}.(K(x_i, x_j))_{i,j}.(\alpha_1, \ldots, \alpha_n)^t = 0$
$\Rightarrow f = 0$. □

3.13 Note. *For $(\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n)$ such as $(K(x_i, x_j))_{i,j}.(\alpha_1, \ldots, \alpha_n)^t = (K(x_i, x_j))_{i,j}.(\beta_1, \ldots, \beta_n)^t$, then $\sum \alpha_i.k_{x_i} = \sum \beta_i.k_{x_i}$*

3.14 Theorem. *Interpolation Theorem in a RKHS*
Let H be a RKHS on a set X. Let $F = \{x_1, \dots, x_n\} \subset X$, $\{\lambda_1, \dots, \lambda_n\} \subset \mathbb{C}$.
There exists $g \in H$ *that interpolates* $\{\lambda_i\}_i$ *on* $\{x_i\}_i \Leftrightarrow (\lambda_1, \dots, \lambda_n) \in Im((K(x_i, x_j))_{i,j})$.
Then, for $(\alpha_1, \dots, \alpha_n)$ *such as* $(K(x_i, x_j))_{i,j}.(\alpha_1, \dots, \alpha_n)^t = (\lambda_1, \dots, \lambda_n)$, $h = \sum_i \alpha_i.k_{x_i}$ *is the function in H of minimal norm that interpolates the values.*
And $\|h\|^2 = \sum_i \overline{\lambda_i}.\alpha_i$.

Proof. If g is a solution, $P_F(g)$ is the solution of minimal norm in H.
$\Rightarrow$
Suppose that we have g a solution. $P_F(g) = \sum_i \beta_i.k_{x_i} \Rightarrow \lambda_i = \sum_j \beta_j.k_{x_j}(x_i)$, $\forall 1 \leq i \leq n$. $\Rightarrow$ $(K(x_i, x_j))_{i,j}.(\beta_1, \dots, \beta_n) = (\lambda_1, \dots, \lambda_n)$.
$\Leftarrow$
If $(\lambda_1, \dots, \lambda_n) \in Im((K(x_i, x_j))_{i,j})$, then for $(\alpha_1, \dots, \alpha_n)$ such as $(K(x_i, x_j))_{i,j}.(\alpha_1, \dots, \alpha_n)^t = (\lambda_1, \dots, \lambda_n)^t$, $h = \sum_i \alpha_i.k_{x_i}$ interpolates the values.

Furthermore $P_F(h) = h$ is the function of minimal norm of the interpolates the values.
And $\|h\|^2 = \sum_{i,j} \overline{\alpha_i}.\alpha_j.K(x_i, x_j) = \overline{(\alpha_1, \dots, \alpha_n)}.(K(x_i, x_j))_{i,j}.(\alpha_1, \dots, \alpha_n)^t = \overline{(\lambda_1, \dots, \lambda_n)}.(\alpha_1, \dots, \alpha_n)^t = \sum_i \overline{\lambda_i}.\alpha_i$. □

3.15 Note. $(K(x_i, x_j))_{i,j}$ *is invertible* $\Leftrightarrow \forall \lambda_1, \dots, \lambda_n \in \mathbb{C}$, $\{\lambda_1, \dots, \lambda_n\}$ *can be interpolated on the* $\{x_i\}_i$ $\Leftrightarrow \{k_{x_i}\}$ *are independant.*

3.16 Theorem. *Let H be a RKHS on X. Let* $f : X \to \mathbb{C}$ *a function.*
Then $f \in H \Leftrightarrow \exists c > 0$ *such as* $c^2 K(x, y) - f(x).\overline{f(y)}$ *is a kernel function on X.*
$\Leftrightarrow \exists c > 0$ *such as for every N, for every* $\{x_1, \dots x_N\}$, $\exists h \in H$ *with* $\|h\| \leq c$ *and* $f(x_i) = h(x_i)$ $\forall 1 \leq i \leq N$.

This theorem gives a necessary and sufficien condition for a function to be in a RKHS only with interpolation by functions in H or with kernel functions, both of them being ponctual conditions.
For Hilbert spaces like Bergman spaces, we have a ponctual critera on every finite subset of Ω that says if a function f is holormorphic and L^2 on Ω.
However, the computation required to verify such a critera isn't easy, even if the reproducing kernel has an explicit formula like for $H^2(\mathbb{D})$ or $B^2(\mathbb{D})$.
Since the reproducing kernels in $B^2(\Omega)$ and $H^2(\Omega)$ have no zeroes, for every set of complex values and for every set of points in Ω, there are functions in these spaces that interpolate these values on the chosen points.
As ploynomials are in $H^2(\mathbb{D})$ and $B^2(\mathbb{D})$, they directly give functions that interpolates a set of values on a set of specific points in the case of the disc, and the unitary maps linking two Bergman spaces (respectively Hardy) generalizes that property for every Ω open and simply connected.
Thus, we had another way to obtain the interpolation properties on Bergman and Hardy spaces, but doing it with the reproducing kernel gives interpolation functions with more properties (like minimizing the norm).

3.17 Theorem. *Aronszajn*
Let H_1, H_2 *be two RKHS on X.*
$H_1 \subset H_2 \Leftrightarrow \exists c > 0$ *such as* $c^2.K_2 - K_1$ *is a kernel function. In this case,* $\|f\|_2 \leq c.\|f\|_1$, $\forall f \in H_1$.

Proof. Only one sub-part of the theorem will be proven, as it illustrates well the importance of being a RKHS.
If $H_1 \subset H_2$, $T : f \in H_1 \mapsto f \in H_2$ is well defined. Let $(f_n)_n \in H_1^{\mathbb{N}}$ who converges towards $f \in H_1$ for $\|.\|_1$ and towards $g \in H_2$ for $\|.\|_2$. So $(f_n, T(f_n))_n$ converges towards (f, g) in $H_1 \times H_2$. As H_1 and H_2 are RKHS, norm convergence implies ponctual convergence. Thus, $\forall x \in X$, $f(x) = \lim_n f_n(x) = g(x)$. So $f = g = T(f)$, which means that the graph of T is closed. By the closed graph Theorem, T is bounded.
Thus, $\forall f \in H_1$, $\|f\|_2 \leq \|T\|.\|f\|_1$. □

3.18 Theorem. *Aronszajn*
Let H_1,H_2 be two RKHS on X.
Then, $k := K_1 + K_2$ is a kernel function on X. Its associated Hilbert space is $H(K) := \{f_1 + f_2,\ f_1 \in H_1,\ f_2 \in H_2\}$, with $\|f\|_H^2 = \inf(\|f_1\|_1^2 + \|f_2\|_2^2$ with $f_1 + f_2 = f)$.
If $H_1 \cap H_2 = \{0\}$, then $\|f_1 + f_2\|_H^2 = \|f_1\|_1^2 + \|f_2\|^2$.

3.19 Note. *Since the condition on K to be a kernel function is that $(K(x, y))$ is self-adjoint positive, the sum of two kernel functions is clearly a kernel fuction.*
Multiplying a kernel function by a $c > 0$ also gives another kernel function.
If the difference of two kernel functions $K_1 - K_2$ is still a kernel function K_3, then $H(K_2) + H(K_3) = H(K_1)$ because $K_2 + K_3 = K_1$.

The next properties deal with composition of a kernel function with another function, in order to go from a set X to another set S.

3.20 Proposition. — *Let K be a kernel function on X, let S be a set, $\varphi : S \to X$ a function. Then $K \circ \varphi$ is a kernel function on S.*

3.21 Theorem. *Let X_1,X_2 be sets, K_1,K_2 kernel functions on X_1,X_2, and $\varphi : X_1 \to X_2$. We have the equivalence :*
i) $\{f \circ \varphi,\ f \in H(K_2)\} \subset H(K_1)$.
ii) $C_\varphi : f \in H(K_2) \mapsto f \circ \varphi \in K(K_1)$ is bounded.
iii) $\exists c > 0$ such as $c^2.K_1 - K_2$ is a kernel function on X_1.

Proof. Proof of i) $\Rightarrow$ ii)
Let $\{f_n\}_n \in H(K_2)^{\mathbb{N}}$ who converges in $\|.\|_2$ norm towards $f \in H(K_2)$, and with $f_n \circ \varphi$ that converges in $\|.\|_1$ norm towards $g \in H(K_1)$. Then, $\forall x \in X_1$, $f(\varphi(x)) = \lim_n(f_n(\varphi(x))) = \lim_n(f_n \circ \varphi(x)) = g(x)$, because convergence in $\|.\|_2$ or $\|.\|_1$ imply ponctual convergence.
Thus, the graph of C_φ is closed, and C_φ is bounded by the closed graph theorem. □

4 Multiplication Operators

4.1 Definition. Let X be a set. Let H be a Hilbert space of functions from X to $\mathbb{C}$.
Let $w : X \to \mathbb{C}$ be a function that verifies $f.w \in H, \forall f \in H$.
We define : $\begin{array}{rccc} M_w: & H & \to & H \\ & f & \mapsto & f.w \end{array}$ the multiplication operator.

On the next part, we will look for functions that give bounded multiplication operators on L^2 spaces and on Hardy and Bergman spaces, and see if these operators can be compact or even better.

4.1 Multiplication operators on $L^2(\Omega)$

Let Ω be an open subset of $\mathbb{C}$.
If $w \in L^\infty(\Omega)$, then $\forall f \in L^2(\Omega)$, $\|f.w\|_{L^2}^2 \leq \|w\|_{L^\infty}^2.\|f\|_{L^2}^2$.
Thus, M_w is a bounded operator.
Furthermore, $\langle g; M_w(f)\rangle = \int_\Omega \overline{g(z)}.f(z).w(z)|dz| = \int_\Omega \overline{g(z).\overline{w(z)}}.f(z)|dz| = \langle M_{\overline{w}}(g); f\rangle$.
Thus, for $L^2(\Omega)$, $M_w^* = M_{\overline{w}}$ and $M_w^*.M_w = M_{|w|^2}$.

4.2 Proposition. — $\sigma(M_w) = \{\lambda \in \mathbb{C}$ *such as* $\forall\varepsilon > 0$, $\mu(w^{-1}(B(\lambda;\varepsilon))) > 0\} = \overline{w(\Omega)}$.
Where $\sigma(M_w)$ *is the spectrum of* M_w, *and* μ *the Lebesgue measure on* $\mathbb{C}$.

Proof. Let $\lambda \in \mathbb{C}$ such as $\forall\varepsilon > 0$, $\mu(w^{-1}(B(\lambda;\varepsilon))) > 0$. We will show that $M_w - \lambda$ isn't invertible.
We note $V_{\lambda,\varepsilon} := w^{-1}(B(\lambda;\varepsilon))$.
Let us fx $\varepsilon > 0$, and take a compact K for which $0 < \mu(V_{\lambda,\varepsilon} \cap K) < \infty$.
Then, $g = \xi_{V_{\lambda,\varepsilon}\cap K}.\frac{1}{\sqrt{\mu(V_{\lambda,\varepsilon}\cap K)}} \in L^2(\Omega)$ and $\|g\|_{L^2} = 1$.
If $M_w - \lambda = M_{w-\lambda}$ is bijective, its inverse is $M_{\frac{1}{w-\lambda}}$.
$\frac{1}{w-\lambda}$ can be correctly defined nearly everywhere as a function from Ω to $\mathbb{C}$ $\{\infty\}$.
But $\|g.\frac{1}{w-\lambda}\|_{L^2}^2 = \int_{V_{\lambda,\varepsilon}\cap K} \frac{1}{\mu(V_{\lambda,\varepsilon}\cap K)}.\frac{1}{|w(z)-\lambda|^2}|dz| \geq \|g\|_{L^2}^2.\frac{1}{\varepsilon^2}$.
Thus, $M_{\frac{1}{w-\lambda}}$ isn't be bounded, so $\lambda \in \sigma(M_w)$.
Conversely, for λ who doesn't verify the property, $\exists\varepsilon > 0$ such as $\mu(\{\text{y with } f(y) \in B(\lambda,\varepsilon)\}) = 0$, the function $\frac{1}{w-\lambda}$ has values in $\mathbb{C}$ nearly everywhere, and $\frac{1}{|w-\lambda|} \leq_{n.e.} \frac{1}{\varepsilon} \Rightarrow \frac{1}{w-\lambda} \in L^\infty(\Omega) \Rightarrow M_{\frac{1}{w-\lambda}}$ is bounded $\Rightarrow \lambda \notin \sigma(M_w)$.

□

4.3 Proposition. — *The eigenvalues of* M_w *are* $\{\lambda \in \mathbb{C}$ *such as* $\mu(w^{-1}(\{\lambda\})) > 0\}$

Proof. If $\mu(w^{-1}(\{\lambda\})) > 0$, $\exists K$ compact such as $0 < \mu(w^{-1}(\{\lambda\}) \cap K) < \infty$.
Thus, $g = \xi_{w^{-1}(\{\lambda\})\cap K} \in L^2(\Omega)$ and $M_w(g) = \lambda.g$. We have an eigenvector for λ.
Conversely, if $\mu(w^{-1}(\{\lambda\})) = 0$, then $M_w(g) = \lambda.g \Leftrightarrow M_w(g) - \lambda.g = 0 \Leftrightarrow (w-\lambda).g = 0 \Leftrightarrow g =_{n.e.} 0$.
Thus, λ has no eigenvectors. □

4.4 Proposition. — *We now have that* $\|M_w\| = \sup\{|\lambda|$ *with* λ *such as* $\forall\varepsilon > 0$, $\mu(w^{-1}(B(\lambda;\varepsilon))) > 0\} = \|w\|_{L^\infty}$.

4.5 Note. *As we explicitly determined the spectrum of multiplication operators on* L^2, *we now know that these are not compact if* $w(\Omega)$ *is not discrete, with maybe a limit point in zero.*
And if we have a $\lambda \in \mathbb{C}$ *with* $\mu(w^{-1}(\{\lambda\})) > 0$, *then* λ *has a space of eigenvectors of infinite dimension.*
Thus, M_w *is compact if and only if* $w \equiv 0$.

4.2 Multiplication operators on $B^2(\Omega)$ and $H^2(\Omega)$

Here, we want that $\forall f \in B^2(\Omega)$, $w.f \in B^2(\Omega)$ ($H^2(\Omega)$ respectively).
For the rest of the section, we will consider the case of $B^2(\Omega)$, but the results are exactly identical for $H^2(\Omega)$.
Since we know that the reproducing kernels or the Bergman and Hardy spaces have no zeroes, the functions $k_z^{B^2(\Omega)} = \overline{K_{B^2}(\Omega)(z;.)}$ also have no zeroes in Ω.
As $k_z^{B^2(\Omega)} \in B^2(\Omega)$, we need $w.k_z^{B^2(\Omega)}$ to be holomorphic on Ω, so $\frac{w.k_z^{B^2(\Omega)}}{k_z^{B^2(\Omega)}} = w$ must be holomorphic on Ω.
The w that make M_w well defined on $B^2(\Omega)$ are way more restricted than on the L^2 case.
We will also take $\|w\|_{L^\infty} < \infty$. Then, $\|M_w\| \leq \|w\|_{L^\infty}$, so M_w is bounded.

4.6 Proposition. — *Let $w : \Omega \to \mathbb{C}$ be in $Hol(\Omega) \cap L^\infty(\Omega)$. Then $M_w : B^2(\Omega) \to B^2(\Omega)$ is well defined, bounded, and , $\sigma(M_w) = \overline{w(\Omega)}$. This also implies that $\|M_w\| = \|w\|_{L^\infty}$.*

Proof. If $\lambda \notin \overline{w(\Omega)}$, then $\frac{1}{|w-\lambda|} \leq \frac{1}{d(\lambda;w(\Omega))} < \infty$. So $\frac{1}{w-\lambda}$ is holomorphic and bounded on Ω, so $M_{w-\lambda}$ is invertible, so $\lambda \notin \sigma(M_w) \Rightarrow \sigma(M_w) \subset \overline{w(\Omega)}$.
Conversely, let $\lambda \in w(\Omega)$. We have $\lambda = w(y)$ for a $y \in \Omega$.
For $f, g \in B^2(\Omega)$, we have $\langle g; M_w(f)\rangle = \int_\Omega \overline{g(z)}.f(z).w(z)|dz| = \int_\Omega \overline{g(z).\overline{w(z)}}.f(z)|dz|$, but $\overline{w}$ isn't holomorphic, so $M_{\overline{w}}$ isn't even well defined.
Thus, contrary to the L^2 case, M_w^* can't be easily expressed.
However, for $g = k_y^{B^2(\Omega)}$, we have : $\langle M_w^*(k_y^{B^2(\Omega)}); f\rangle = \langle k_y^{B^2(\Omega)}; M_w(f)\rangle = w(y).f(y) = w(y).\langle k_y^{B^2(\Omega)}; f\rangle = \rangle \overline{w(y)}.k_y^{B^2(\Omega)}; f\rangle$.
So $\rangle M_w^*(k_y^{B^2(\Omega)}) - \overline{w(y)}.k_y^{B^2(\Omega)}; f\rangle = 0, \forall f \in B^2(\Omega)$.
Thus, $M_w^*(k_y^{B^2(\Omega)}) = \overline{w(y)}.k_y^{B^2(\Omega)}$, so $\overline{w(y)}$ is an eigenvalue of $M_w^* \Rightarrow w(y) \in \{\overline{\lambda}, \lambda \in \sigma(M_w^*)\} = \sigma(M_w)$ $\Rightarrow w(\Omega) \subset \sigma(M_w) \Rightarrow \overline{w(\Omega)} \subset \sigma(M_w)$. □

4.7 Note. *We saw that the spectrum of multiplication operators on B^2 and H^2 is the same as on L^2. Thus, M_w is compact if and only if $w \equiv 0$.*

5 Composition Operators

5.1 Definition. - Let X_1, X_2 be sets, and F_1, F_2 be vector spaces of complex-valued functions over X_1, X_2. Let $\phi : X_1 \to X_2$.
If $\forall f \in F_2$, $f \circ \phi \in F_1$, we define : $\begin{array}{cccc} C_\phi : & F_2 & \to & F_1 \\ & f & \mapsto & f \circ \phi \end{array}$ the **composition operator**.
- Let $w : X_1 \to \mathbb{C}$ be a function that verifies $f.w \in F_1, \forall f \in F_1$.
We define $M_w \circ C_\phi : f \in F_2 \mapsto (f \circ \phi).w \in F_1$ the **weighted composition operator**.

Before studying composition operators on Hardy and Bergman spaces, we will study composition operators on spaces holomorphic functions with the topology of the uniform convergence on every compact.
We will also only focus on $\mathbb{D}$ for now, with a study of the biholomorphisms $\mathbb{D} \to \mathbb{D}$, in order to have lighter proofs.
Properties of composition operators on Bergman/Hardy spaces of a space Ω will be discussed later on, as the unitary map that links two Bergman/Hardy spaces doesn't send a composition operator to another composition operator, but to a weighted composition operator.

5.1 Composition operators on Hol($\mathbb{D}$)

We are looking at $\phi : \mathbb{D} \to \mathbb{D}$. Since Hol($\mathbb{D}$) contains holomorphic functions with no zeroes, C_ϕ will be well defined on Hol($\mathbb{D}$) if and only if ϕ is holomorphic.

5.2 Proposition. — *If ϕ isn't constant, then C_ϕ is injective.*

Proof. $C_\phi f = C_\phi g \Leftrightarrow f = g$ on $\phi(\mathbb{D}) \Leftrightarrow f = g$ on $\mathbb{D}$ by of the isolated zeroes theorem. □

5.3 Note. *For $C^0(\mathbb{D})$ and $\phi : \mathbb{D} \to \mathbb{D}$ continuous, C_ϕ is injective if and only if $\phi(\mathbb{D}) = \mathbb{D}$.*

5.4 Theorem. *C_ϕ is bijective on Hol($\mathbb{D}$) if and only if ϕ is bijective.*

Proof. $\Leftarrow$
If ϕ is bijective, ϕ^{-1} is holomorphic, and $(C_\phi)^{-1} = C_{\phi^{-1}}$.
$\Rightarrow$
If ϕ isn't injective, we have α, β such as $\phi(\alpha) = \phi(\beta)$. Thus, $\forall f \in Hol(\mathbb{D})$, $f \circ \phi(\alpha) = f \circ \phi(\beta) \Rightarrow C_\phi$ isn't surjective $\Rightarrow C_\phi$ isn't invertible on Hol($\mathbb{D}$), contradiction.
If ϕ isn't surjective, there exists a $w \in \mathbb{D} - \phi(\mathbb{D})$.
Let $f : z \mapsto \frac{1}{z-w}$ and $g : z \mapsto \frac{1}{\phi(z)-w}$. f is holomorphic on $\phi(\mathbb{D})$ but not on $\mathbb{D}$, g is holomorphic on $\mathbb{D}$.
Since C_ϕ is surjective, we would have $h \in Hol(\mathbb{D})$ such as $h \circ \phi(z) = g(z) = f \circ \phi(z) \Rightarrow h = f$ on $\phi(\mathbb{D})$ $\Rightarrow h \equiv f \Rightarrow h \notin Hol(\mathbb{D})$, contradiction. □

5.5 Theorem. *The biholomorphisms from $\mathbb{D}$ to $\mathbb{D}$ are of the form : $e^{i\theta}.\varphi_\alpha$, with $\varphi_\alpha : z \mapsto \frac{z-\alpha}{\overline{\alpha}.z-1}$, for $\alpha \in \mathbb{D}$, $\theta \in [0, 2\pi[$.*
Such a biholomorphism is determined by fixing the image of two points.

5.6 Note. *We have $\varphi_\alpha(\alpha) = 0$, $\varphi_\alpha(0) = \alpha$, and $\varphi_\alpha^{-1} = \varphi_\alpha$.*

5.7 Theorem. *Let $\phi : \mathbb{D} \to \mathbb{D}$, holomorphic, with $\phi(0) = 0$. If ϕ isn't a rotation, then $\phi_n := \phi \circ \ldots \circ \phi \to 0$ uniformly on every compact.*

Proof. For $g : z \mapsto \left| \begin{array}{l} \frac{\phi(z)}{z} \text{ if } z \neq 0 \\ \phi'(0) \text{ else} \end{array} \right.$, $g \in Hol(\mathbb{D})$ and $\sup_{\mathbb{D}}(|g|) = lim_{r \to 1^-}(sup_{z=r.e^{i\theta}}|g(z)|) \leq 1$, by applying the maximum principle.
So $|\phi(z)| \leq |z|$, $\forall z \in \mathbb{D}$. Because ϕ isn't a rotation, we can't have equality in the maximum principle inequality, so $|\phi(z)| < |z|$, $\forall z \in \mathbb{D}$.
Let's fix K a compact. We have $0 < r < 1$ such as $K \subset B(0, r)$. Thus, $|\phi(z)| < r$ on $\partial B(0, r)$, so $M_r := sup_{\partial B(0,r)}|\phi| < r$.
For $\psi : z \mapsto \frac{\phi(r.z)}{M_r}$, $\psi \in Hol(\mathbb{D})$, $\psi(0) = 0$, and $\psi(\mathbb{D}) \subset \mathbb{D}$. So, $|\psi(z)| < |z|$ with the previous argument.
Thus, $|\phi(z)| \leq \frac{M_r}{r}.|z|$, with $\frac{M_r}{r} < 1$. So $|\phi_n(z)| \leq (\frac{M_r}{r})^n.|z|$, which means that ϕ_n converges to 0 uniformly on K. □

5.8 Corollary. *Let $\phi : \mathbb{D} \to \mathbb{D}$, holomorphic, non-bijective, with $\phi(\alpha) = \alpha$ for an $\alpha \in \mathbb{D}$. Then $\phi_n := \phi \circ \ldots \circ \phi \to \alpha$ uniformly on every compact.*
By looking at $\psi := \varphi_\alpha^{-1} \circ \phi \circ \varphi_\alpha$, we can apply the theorem to ψ and obtain the conclusion on ϕ.
This also says that a non-bijective holomorphic function of $\mathbb{D} \to \mathbb{D}$ has at most 1 fix point.

5.9 Theorem. *Theorem of Koenigs*
Let $\phi : \mathbb{D} \to \mathbb{D}$, holomorphic, non-bijective, with a fix point $\alpha \in \mathbb{D}$ such as $\phi'(\alpha) \neq 0$.
Then, the eigenvalues of C_ϕ on Hol($\mathbb{D}$) are $\phi'(\alpha)^n$, $\forall n \geq 0$.
These eigenvalues are of multiplicity one. For σ an eigenfunction of $\phi'(\alpha)$, σ^n is an eigenfunction of $\phi'(\alpha)^n$.
If ϕ is injective, then σ is injective.

Proof. By looking at $\varphi_\alpha^{-1} \circ \phi \circ \varphi_\alpha$, we will suppose that $\alpha = 0$. All the results for $\alpha \in \mathbb{D}$ can then be obtained by composing again by φ_α.
– Since $\phi'(0) \neq 0$, ϕ isn't constant. Thus, 0 isn't an eigenvalue.
– Let λ be an eigenvalue with a constant eigenfunction f. Then $f \circ \phi = f = \lambda.f$, so $\lambda = 1$.
– Let λ be an eigenvalue with a non-constant eigenfunction f. Then $\lambda \neq 1$ and $f(0) = 0$.
By applying theorem 5.7, ϕ_n converges towards 0 uniformly on every compact.
If λ was 1, then $f \circ \phi = f \Rightarrow f \circ \phi_n = f \Rightarrow f = \lim(f \circ \phi_n) = f(0)$, by uniform convergence $\Rightarrow$ f is constant, contradiction.
Also, $f \circ \phi(0) = f(0) = \lambda.f(0)$. Since $\lambda \neq 1$, we have $f(0) = 0$.

– λ is of the form $\phi'(0)^n$, for a $n \geq 0$.
Let f be an eigenfunction for λ. We have a $n \geq 0$ for which $f(z) = \sum_{k \geq n} a_k.z^k$, with $a_n \neq 0$.
Thus, for $z \neq 0$, we obtain : $\lambda = \frac{f(\phi(z))}{f(z)} = (\frac{\phi(z)}{z})^n.\frac{a_n + a_{n+1}.\phi(z) + \dots}{a_n + a_{n+1}.z + \dots} \rightarrow_{z \to 0} \phi'(0)^n$.
– The multiplicity of the eigenvalues is 1.
If $\lambda = 1$, then an eigenfunction f can only be constant.
Let f be an eigenfunction for $\lambda \neq 1$. Then $f(0) = 0$, and $\exists N > 0$ such as $\lambda = \phi'(0)^N$.
We also saw from the previous point that $f(z) = \sum_{n \geq N} a_n.z^n$, with $a_N \neq 0$. So $f'(0) = \dots = f^{(N-1)}(0) = 0$ and $f^{(N)}(0) \neq 0$. By derivating $f \circ \phi = \lambda.f$ multiple times and evaluating in $z = 0$, we can see by induction that for any $m > N$, $f^{(m)}(0)$ depends on $f^{(N)}(0), \dots, f^{(m-1)}(0)$ and on $\phi(0), \phi'(0), \dots, \phi^{(m)}(0)$.
Since $\phi(0), \phi'(0), \dots, \phi^{(m)}(0)$ are already known, we can see that $f^{(m)}(0)$ will be determined by $f^{(N)}(0)$. Since $f(z) = \sum_{n \geq N} \frac{f^{(n)}}{n!}.z^n$, f is determined by $f^{(N)}(0)$.
Thus, for f,g eigenfunctions such as $f^{(N)}(0) = g^{(N)}(0)$, we have $f = g$.
– $\exists \sigma \in Hol(\mathbb{D})$ such as $\sigma \circ \phi = \phi'(0).\sigma$.
Let $\lambda = \phi'(0)$. We define $\sigma_n := \lambda^{-n}.\phi_n$. Thus, $\sigma_n \circ \phi = \lambda.\sigma_{n+1}$.
We have : $\sigma_n(z) = z.\frac{\phi(z)}{\lambda.z}.\frac{\phi_2(z)}{\lambda.\phi(z)} \dots \frac{\phi_n(z)}{\lambda.\phi_{n-1}(z)} = z.\Pi_{i=0}^{n-1} F(\phi_i(z))$, with $F(z) = \frac{\phi(z)}{\lambda.z}$.
We will show that $\Pi_{j \geq 0} F(\phi_j(z))$ converges uniformly on every compact.
We have $\|1 - F\|_{L^\infty} \leq 1 + \|F\|_{L^\infty} \leq 1 + \frac{1}{|\lambda|} := A$. And $F(0) = \frac{\phi'(0)}{\lambda} = 1$.
Thus, $\frac{|1-F(z)|}{A} \leq |z|, \forall z \in \mathbb{D}$. Let's fix $0 < r < 1$. We know that we have $|\phi_n| \leq (\frac{M_r}{r})^n.|z|$, with $\frac{M_r}{r} < 1$.
This means that : $|1 - F(\phi_n(z)| \leq A.|\phi_n(z)| \leq A.(\frac{M_r}{r})^n.|z|$.
So $\sum_{j \geq 0} |1 - F(\phi_j(z)|$ converges uniformly on every compact of $\mathbb{D}$. Thus, $\Pi_{j \geq 0} F(\phi_i(z))$ also converges uniformly on every compact of $\mathbb{D}$, and its limit is holomorphic.
We obtain then $\sigma \in Hol(\mathbb{D})$ such as $\sigma \circ \phi = \lambda.\sigma$.
– σ^n is an eigenvalue for $\phi'(0)^n$.
We have $\sigma^n \circ \phi = (\sigma \circ \phi)^n = (\phi'(0).\sigma)^n = \phi'(0)^n.\sigma^n$.
– If ϕ is injective, then σ is injective.
ϕ injective $\Rightarrow$ ϕ_n is injective $\Rightarrow$ σ_n is injective $\Rightarrow$ σ is either injective or constant $\Rightarrow$ σ is injective, because σ is non-constant.

□

5.10 Note. *This theorem gives a really detailed description of the eigenvalues of a composition operator in Hol($\mathbb{D}$). It will also be helpful for composition operators on Hardy or Bergman spaces, as eigenvalues of an operator on these spaces are also eigenvalues of this operator in Hol($\mathbb{D}$).*
So as long as the function ϕ has a fix point, we have a clear description of them.
Koenig's theorem can be extended to Hol(Ω), for Ω an open and simply connected space of $\mathbb{C}$, by composing $\phi : \Omega \to \Omega$, holomorphic, with $\psi : \Omega \to \mathbb{C}$ a biholomorphism that sends the fix point of ϕ to 0.

5.2 Bounded composition operators on $H^2(\mathbb{D})$

The previous theorems gave many information about composition operators on Hol($\mathbb{D}$).
Theorem 3.21 says that a composition operator C_ϕ on a Bergman or Hardy space is bounded if and only if it is well defined.
But as of now, we don't have any property ensuring that C_ϕ is well defined on these spaces.
The goal of this subsection is to study definiteness and boundedness of such composition operators.

5.11 Theorem. *Littlewood's subordination theorem*
Let $\phi : \mathbb{D} \to \mathbb{D}$, holomorphic, with $\phi(0) = 0$. Then $\forall f \in H^2(\mathbb{D})$, $C_\phi f \in H^2(\mathbb{D})$ and $\|C_\phi\| \leq 1$.

To prove this theorem, we will need to use a result about harmonic functions.

5.12 Definition. Let $g : \mathbb{D} \to \mathbb{R}$ a continuous function. g is called a **harmonic** function on $\mathbb{D}$ if and only if g verifies the mean value property : $\forall z \in \mathbb{D}, \forall 0 < r < 1 - |z|, g(z) = \frac{1}{2\pi}. \int_0^{2\pi} g(z + r.e^{i\theta})d\theta$.
g is called a **subharmonic** function on $\mathbb{D}$ if and only if : $\forall z \in \mathbb{D}, \forall 0 < r < 1 - |z|, g(z) \leq \frac{1}{2\pi}. \int_0^{2\pi} g(z + r.e^{i\theta})d\theta$

5.13 Theorem. *Let $g : \mathbb{D} \to \mathbb{R}$ be a continuous function.*
If g is harmonic on $\mathbb{D}$, then g is analytic on $\mathbb{D}$.
If two harmonic functions on $\mathbb{D}$, g_1 and g_2 are equal on an open space U, then $g_1 \equiv g_2$.
g is harmonic on $\mathbb{D}$ if and only if it can be expressed as the real part of an holomorphic function in Hol($\mathbb{D}$).
Let $z_0 \in \mathbb{D}$ and $0 < r < 1 - |z_0|$. Let $h : \partial B(z_o, r) \to \mathbb{R}$ be a continuous function. Then, there exists an unique 1function $g : \overline{B(z_o, r)} \to \mathbb{R}$, harmonic on $B(z_0, r)$, such as $g \equiv h$ on $\partial B(z_o, r)$.

With this theorem, we can now prove Littlewood's subordination theorem.

Proof. Using Schwarz's lemma like beore, we obtain : $|\phi(z)| \leq |z|, \forall z \in \mathbb{D}$. We also note that $\forall f \in H^2(\mathbb{D})$, f is holomorphic, and f^2 is holomorphic.
Thus, the mean value property applied to f^2 gives : $\forall z \in \mathbb{D}, \forall 0 < r < 1 - |z|$,

$$f^2(z) = \frac{1}{2\pi}. \int_0^{2\pi} f^2(z + r.e^{i\theta})d\theta$$

$$\Rightarrow |f^2(z)| \leq \frac{1}{2\pi}. \int_0^{2\pi} |f^2(z + r.e^{i\theta})|d\theta$$

Therefore, $|f^2|$ is subharmonic on $\mathbb{D}$.
Let's fix $0 < r < 1$. Since $|f^2|$ is continuous from $\partial B(0, r)$ to $\mathbb{R}_+$, theorem 5.13 gives us an unique $h : \overline{B(0, r)} \to \mathbb{R}$, h harmonic on $B(0, r)$, such as $h \equiv |f^2|$ on $\partial B(0, r)$.
Since $|f^2| \geq 0$ on $\partial B(0, r)$, $h \geq 0$ on $B(0, r)$ by harmoniticity.
Since $|f^2|$ is subharmonic, we also obtain that : $\forall z \in \overline{B(0, r)}, |f^2(z)| \leq h(z) \Rightarrow |f(\phi(z))|^2 \leq h(\phi(z))$, because $|\phi(z)| \leq |z|$.
Thus, we finally obtain : $\frac{1}{2\pi}. \int_0^{2\pi} |f \circ \phi(r.e^{i\theta})|^2 d\theta \leq \frac{1}{2\pi}. \int_0^{2\pi} h \circ \phi(r.e^{i\theta})d\theta = h \circ \phi(0) = h(0) = \frac{1}{2\pi}. \int_0^{2\pi} h(r.e^{i\theta})d\theta = \frac{1}{2\pi}. \int_0^{2\pi} |f(r.e^{i\theta})|^2 d\theta$

$$\Rightarrow \|f \circ \phi\|^2_{H^2(\mathbb{D})} = \lim_{r \to 1^-} (\frac{1}{2\pi}. \int_0^{2\pi} |f \circ \phi(r.e^{i\theta})|^2 d\theta) \leq \lim_{r \to 1^-} (\frac{1}{2\pi}. \int_0^{2\pi} |f(r.e^{i\theta})|^2 d\theta) = \|f\|^2_{H^2(\mathbb{D})}$$

$\Rightarrow C_\phi$ is well defined on $H^2(\mathbb{D})$ and $\|C_\phi\| \leq 1$.

□

We will now prove a little lemma before coming to the main theorem of this subsection.

5.14 Lemma. *For $\varphi_\alpha : z \mapsto \frac{z - \alpha}{\overline{\alpha}.z - 1}$, $\alpha \in \mathbb{D}$, C_{φ_α} is bounded on $H^2(\mathbb{D})$ and $\|C_{\varphi_\alpha}\| \leq \sqrt{\frac{1+|\alpha|}{1-|\alpha|}}$.*

Proof. Let $0 < r < 1$.
$\int_0^{2\pi} |f \circ \varphi_\alpha(r.e^{i\theta})|^2 d\theta = \int_0^{2\pi} |f(r.e^{it})|^2.|\varphi'_\alpha(r.e^{it})|dt \leq \int_0^{2\pi} |f(r.e^{it})|^2.\frac{1-(r.|\alpha|)^2}{|1-\overline{\alpha}.r.e^{it}|^2}dt$
$\leq \int_0^{2\pi} |f(r.e^{it})|^2.\frac{1-(r.|\alpha|)^2}{(1-r.|\alpha|)^2}dt \leq \int_0^{2\pi} |f(r.e^{it})|^2.\frac{1+r.|\alpha|}{1-r.|\alpha|}dt \leq \int_0^{2\pi} |f(r.e^{it})|^2.\frac{1+|\alpha|}{1-|\alpha|}dt$
Because for $g(r) = \frac{1+r.|\alpha|}{1-r.|\alpha|}$, $g(r) = \frac{2}{1-r.|\alpha|} - 1$, so $g(r) \leq g(1) = \frac{1+|\alpha|}{1-|\alpha|}$.
Thus, $\|f \circ \varphi_\alpha\|^2_{H^2(\mathbb{D})} \leq \frac{1+|\alpha|}{1-|\alpha|}.\|f\|^2_{H^2(\mathbb{D})}$. □

5.15 Theorem. *Let $\phi : \mathbb{D} \to \mathbb{D}$ holomorphic. Then C_ϕ is bounded on $H^2(\mathbb{D})$ and $\|C_\phi\| \leq \sqrt{\frac{1+|\phi(0)|}{1-|\phi(0)|}}$.*

Proof. For $\psi = \varphi_{\phi(0)} \circ \phi$, we have $\psi(0) = 0$, so $\|C_\psi\| \leq 1$, and $\phi = \varphi_{\phi(0)} \circ \psi$, because $\varphi^{-1}_{\phi(0)} = \varphi_{\phi(0)}$.
Thus, $\|C_\phi\| \leq 1.\sqrt{\frac{1+|\phi(0)|}{1-|\phi(0)|}}$.

□

See [4], Lectures on composition operators and analytic function theory, Ch 1-4.

5.3 Bounded composition operators on $B^2(\mathbb{D})$

The same theorems can be proven for $B^2(\mathbb{D})$. The proofs use the exact same arguments, the only difference being in the expression of the $H^2(\mathbb{D})$ norm with the $B^2(\mathbb{D})$ norm.

5.16 Theorem. *Littlevood's subordination theorem Let $\phi : \mathbb{D} \to \mathbb{D}$, holomorphic, with $\phi(0) = 0$. Then $\forall f \in B^2(\mathbb{D})$, $C_\phi f \in B^2(\mathbb{D})$ and $\|C_\phi\| \leq 1$.*

5.17 Lemma. *For $\varphi_\alpha : z \mapsto \frac{z-\alpha}{\overline{\alpha}.z-1}$, $\alpha \in \mathbb{D}$, C_{φ_α} is bounded on $B^2(\mathbb{D})$ $\|C_{\varphi_\alpha}\| \leq \sqrt{\frac{1+|\alpha|}{1-|\alpha|}}$.*

5.18 Theorem. *Let $\phi : \mathbb{D} \to \mathbb{D}$ holomorphic. Then C_ϕ is bounded on $B^2(\mathbb{D})$ and $\|C_\phi\| \leq \sqrt{\frac{1+|\phi(0)|}{1-|\phi(0)|}}$.*

5.19 Note. *A simpler proof for the Hardy case can be done by using an operator called the backwards shift operator to show that C_ϕ is well defined and bounded over polynomials, and then use the density of the polynomials to get the result on the whole space.*
However, this proof doesn't extend to the Bergman case easily because of the difference of orthonormal basis : $\{z^n\}_n$ for one side and $\{z^n.\sqrt{\frac{\pi}{n+1}}\}_n$ on the other side.

5.4 Bounded composition operators on $B^2(\Omega), H^2(\Omega)$

5.20 Note. *For Ω an open and simply connected space, $\psi : \mathbb{D} \to \Omega$ a biholomorphism, $\psi'^p : \mathbb{D} \to \mathbb{C}$ can be well defined for any $p > 0$, and $U : f \in B^2(\Omega) \mapsto (f \circ \psi).(\psi') \in B^2(\mathbb{D})$ and $V : f \in H^2(\Omega) \mapsto (f \circ \psi).(\psi')^{\frac{1}{2}} \in H^2(\mathbb{D})$ are unitary maps.*
For $\varphi : \Omega \to \Omega$, we can study the boundedness or compactness of C_φ on $B^2(\Omega)$ (resp $H^2(\Omega)$) by studying $UC_\varphi U^{-1}$ on $B^2(\mathbb{D})$ (resp $VC_\varphi V^{-1}$ on $H^2(\mathbb{D})$).
For $\phi := \psi^{-1} \circ \varphi \circ \psi$, we have : $A_{B^2,\varphi} := UC_\varphi U^{-1} = M_{\frac{\psi'}{\psi' \circ \phi}} C_\phi$ and $A_{H^2,\varphi} := VC_\varphi V^{-1} = M_{(\frac{\psi'}{\psi' \circ \phi})^{\frac{1}{2}}} C_\phi$.
As said in the beginning of the section, $A_{B^2,\varphi}$ and $A_{H^2,\varphi}$ are weighted composition operators. We know from theorems 5.15 and 5.18 that C_ϕ is bounded on $B^2(\mathbb{D})$ and $H^2(\mathbb{D})$, but the multiplication part may not be bounded depending on the shape of Ω.
A weighted composition operator can be bounded (even compact) even if its multiplication part is not bounded, but this is not always true.
Thus, the result obtained on $\mathbb{D}$ that every composition operator is bounded is not true for every open and simply connex space Ω.

Now that we possess two families of operators that are well defined and bounded over Hardy and Bergman spaces, namely the multiplication and composition operators, we will recall some theorems for bounded operators on Hilbert spaces in order to study the potential compactness of such operators. The section following these recalls will focus on approximation numbers, elements that help in giving interesting properties about compactness and compact operators.

6 Bounded Operators on Hilbert spaces

For most of the theorems, we will suppose that H is a complex and separable Hilbert space. Banach spaces will be used for some general properties, but the objective is to obtain general theorems on bounded operators on complex and separable Hilbert spaces.

6.1 Bounded Operators and Polar decomposition theorem

6.1 Definition. Let E,F be Banach spaces.
We denote by $\mathcal{L}(E, F)$ the set of bounded operators from E to F. If E=F, we note that space $\mathcal{L}(E)$.
Let H be an Hilbert space with its inner product $\langle ; \rangle_H$.
Let $A \in \mathcal{L}(H)$. By applying Riesz Lemma, we can show that A possesses an adjoint operator $A^* \in \mathcal{L}(H)$, such as : $\forall f, g \in H$, $\langle f; A.g\rangle_H = \langle A^*.f; g\rangle_H$.
An operator $A \in \mathcal{L}(H)$ is self-adjoint if and only if $\forall f, g \in H$, $\langle f; A.g\rangle_H = \langle A.f; g\rangle_H \Leftrightarrow A^* = A$.
An operator $A \in \mathcal{L}(H)$ is positive if and only if $\forall f \in H$, $\langle f; A.f\rangle_H \geq 0$. A is definite if and only if $\forall f \in H$, $f \neq 0$, $\langle f; A.f\rangle_H \neq 0$.

6.2 Note. *$\forall A \in \mathcal{L}(H)$, A^*A is self-adjoint positive.*

6.3 Proposition. — *Every positive operator on a complex Hilbert space is self-adjoint.*

As we will only be working with complex Hilbert spaces, this property will always be usable.

6.4 Theorem. *Let H be a separable Hilbert space. Then H possesses an orthonormal basis.*

As we will only be working with separable Hilbert spaces, this theorem will always be usable.

6.5 Theorem. *Let H be a Hilbert space, let $A \in \mathcal{L}(H)$, A positive.
Then $\exists! B \in \mathcal{L}(H)$, B positive, that verifies $B^2 = A$.
This operator is noted by $\sqrt{A} := B$.
For any $C \in \mathcal{L}(H)$ that commutes with A, C will commute with B.
If A is self-adjoint, so i $\sqrt{A}$.*

6.6 Definition. Let $A \in \mathcal{L}(H)$. We note $|A| := \sqrt{A^*A}$.

6.7 Note. *It is in general false that $|AB| = |A||B|$, $|A^*| = |A|$, or $|A + B| \leq |A| + |B|$.
However, $|A|$ is self-adjoint positive.
For any $f \in H$, $\||A|.f\|_H^2 = \langle |A|.f; |A|.f\rangle_H = \langle f; |A|^2.f\rangle_H = \langle f; A^*A.f\rangle_H = \|A.f\|^2$, and $\||A|\| = \|A\|$.*

6.8 Definition. Let $U \in \mathcal{L}(H)$. U is a **partial isometry** if $U|_{Ker(U)^\perp}$ is an isometry from $Ker(U)^\perp$ to $U(H)$.

6.9 Theorem. *Polar decomposition theorem
Let $A \in \mathcal{L}(H)$. Then $\exists U$ a partial isometry such as $A = U.|A|$.
U is unique if we choose $Ker(U) = Ker(A)$. Moreover, $U(H) = \overline{A(H)}$.*

Proof. We define $U : |A|(H) \to A(H)$ by $U(|A|.f) = A.f$.
We have : $\||A|.f\|_H^2 = \|A.f\|_H^2$, and $|A|.f = 0 \Leftrightarrow A.f = 0$ so U is well defined and isometric. It extends to an isometry from $\overline{|A|(H)}$ to $\overline{A(H)}$.
We extend U on H by $U|_{\overline{|A|(H)}^\perp} = 0$.
Since $|A|$ is self-adjoint, $Ker(|A|) = Im(|A|)^\perp$, and $|A|.f = 0 \Leftrightarrow A.f = 0$, $Ker(A) = Ker(|A|) = Ker(U)$. ☐

See [8], Methods of modern mathematical physics, I : Functional analysis, Ch VI.4-VI.6.

6.2 Compact Operators and Hilbert-Schmidt theorem

6.10 Definition. Let X,Y be Banach spaces.
An operator $A \in \mathcal{L}(X,Y)$ is **compact** if and only if the image of the unit ball by A is relatively compact.
The subspace of compact operators is noted $K(X,Y)$.

6.11 Note. *Finite rank operators are compact.*

6.12 Definition. A sequence $\{x_n\}_n \in H^{\mathbb{N}}$ is **weakly convergent** if and only if $\{\langle y; x_n\rangle_H\}_n$ is convergent $\forall y \in H$.
In a Banach space setting, $\{x_n\}_n$ is weakly convergent if and only if $\{\mathcal{L}(x_n)\}_n$ is convergent $\forall l \in H^*$.

6.13 Proposition. *Let B be a compact operator. Then, $\forall \{x_n\}_n$ weakly convergent, $\{B.x_n\}_n$ is convergent.*

6.14 Proposition. *Let X,Y be Banach spaces. Let $T \in \mathcal{L}(X,Y)$. We have :*
i) If $\{T_n\}$ is a sequence of compact operators that converge in norm towards T, then T is compact. Thus, $K(X,Y)$ is closed n $\mathcal{L}(X,Y)$.
ii) For Z another Banach space, $S \in \mathcal{L}(Y,Z)$, if T or S is compact, then ST is compact.
iii) If X,Y are Hilbert spaces, T is compact if and only if T^* is compact.

6.15 Theorem. *Every compact operator on a separable Hilbert space is the norm limit of a sequence of finite rank operators.*

Proof. Let $\{\psi_j\}_j$ be an orthonormal basis of H. We define $\lambda_n := \sup_{f \in Span(\psi_1,\ldots,\psi_n)^\perp, \|f\|=1}(\|Tf\|)$.
We remark that $\{\lambda_n\}_n$ is monotone decreasing.
For any $n \geq 0$, let $f_n \in Span(\psi_1,\ldots,\psi_n)^\perp$, $\|f_n\| = 1$ with $\|Tf_n\| \geq \frac{\lambda_n}{2}$.
The sequence $\{f_n\}_n$ weakly converges towards 0. Since T is compact, $\{Tf_n\}_n$ converges towards 0. Thus, λ_n converges towards 0.
Therefore, $\sum_{n=1}^{N} \langle \psi_n; .\rangle.(T\psi_n)$ are finite rank operators and they converge in operator norm towards T, as the norm of the difference is λ_n. □

6.16 Proposition. *- If A is a compact operator on $\mathcal{L}(H)$, then either $(I-A)^{-1}$ exists, or $A\Psi = \Psi$ has a solution.*
- If A is a compact operator, every λ in $\sigma(A) - \{0\}$ is an eigenvalue of A.

6.17 Theorem. *Riesz-Schauder theorem*
Let A be a compact operator on H. Then $\sigma(A)$ is discrete, with no limit point except sometimes 0.
$\forall \lambda \in \sigma(A)$, $\lambda \neq 0$, λ is an eigenvalue of A of finite multiplicity.

6.18 Theorem. *Hilbert-Schmidt theorem*
Let $A \in \mathcal{L}(H)$ be self-adjoint compact.
Then there is $\{\Psi_n\}_n$ an orthonormal set of H and $\{\lambda_n\}_n$ a sequence of real values with $\lambda_n \geq 0$, $\lambda_{n+1} \leq \lambda_n$, $\lambda_n \to 0$, such as :

$$A = \sum_{n \geq 0} \lambda_n.\langle \Psi_n; .\rangle_H.\Psi_n$$

6.19 Theorem. *Canonical form of compact operators*
Let $A \in \mathcal{L}(H)$ be compact.
Then there are $\{\Psi_n\}_n$ and $\{\Phi_n\}_n$ two orthonormal sets of H, and $\{\lambda_n\}_n$ a sequence of real values with $\lambda_n \geq 0$, $\lambda_{n+1} \leq \lambda_n$, $\lambda_n \to 0$, such as :

$$A = \sum_{n \geq 0} \lambda_n.\langle \Psi_n; .\rangle_H.\Phi_n$$

6.20 Note. *In the last theorem, the non-zero λ_n are the non-zero eigenvalues of $|A|$, and $|A| = \sum_{n\geq 0} \lambda_n.\langle \Psi_n; .\rangle_H.\Psi_n$.*
*These non-zero λ_n are called the **singular values** of A.*
If A is self-adjoined, a singular value of A is the modulus of an eigenvalue of A.
However, there is no such explicit relation between singular values and eigenvalues for a general bounded operator A.

Now that the main properties of compact operators have been put together, we can calmly start to define approximation numbers and study their properties. Many types of compact operators will also appear in the way.

7 Approximation numbers and $\mathcal{I}_p$ Ideals

7.1 Approximation numbers

7.1 Definition. For $r \geq 0$, E,F Banach spaces, and $T \in \mathcal{L}(E, F)$, we define $\alpha_r(T) := inf\{\|T - A\|,\ A \in \mathcal{L}(E, F),\ rank(A) \leq r\}$, the **r-th approximation number** of T.

We clearly have : $\|T\| = \alpha_0(T) \geq \alpha_1(T) \geq \alpha_2(T) \geq \ldots \geq 0$.
Many other properties can be found for these approximation numbers, that make them easier to manipulate.

7.2 Proposition. — *For $S, T \in \mathcal{L}(E, F)$, $r, s \geq 0$, we have :*

$$\alpha_{r+s}(S + T) \leq \alpha_r(S) + \alpha_s(T)$$

Proof. Let $\varepsilon > 0$. We choose $A, B \in \mathcal{L}(E, F)$, $rank(A) \leq r$, $rank(B) \leq s$, such as $\|S - A\| \leq \alpha_r(S) + \varepsilon$, $\|T - B\| \leq \alpha_s(T) + \varepsilon$.
Thus, $rank(A + B) \leq r + s$, and $\|(S + T) - (A + B)\| \leq \alpha_r(S) + \alpha_s(T) + 2\varepsilon$.
$\Rightarrow \alpha_{r+s}(S + T) \leq \alpha_r(S) + \alpha_s(T)$. □

7.3 Proposition. — $\forall r \geq 0$, $\forall S, T \in \mathcal{L}(E, F)$,

$$|\alpha_r(S) - \alpha_s(T)| \leq \|S - T\|$$

Proof. We have : $\alpha_r(S) \leq \alpha_r(T) + \alpha_0(S - T) \Rightarrow \alpha_r(S) - \alpha_s(T) \leq \|S - T\|$. □

7.4 Proposition. — $\forall r \geq 0$, $\forall \lambda \in \mathbb{C}$, $\forall T \in \mathcal{L}(E, F)$,

$$\alpha_r(\lambda.T) = |\lambda|\alpha_r(T)$$

Proof. If $\lambda = 0$, the property is verified.
If $\lambda \neq 0$, we have : $\alpha_r(\lambda.T) = inf\{\|\lambda.T - A\|, rank(A) \leq r\} = |\lambda|.inf\{\|T - \frac{A}{\lambda}\|, rank(A) \leq r\} = |\lambda|\alpha_r(T)$. □

See [1], Nuclear locally convex spaces, Ch 8.

7.5 Proposition. — *If $\exists r \geq 0$ such as $\alpha_r(T) = 0$, then $rank(T) \leq r$.*

Proof. Suppose that $rank(T) > r$. Then we have $x_1, \dots, x_{r+1}$ such as $Tx_1, \dots Tx_{r+1}$ are linearly independant.
This gives r+1 linear forms $b_k \in F'$ such as $\langle b_k; Tx_i; \rangle := b_k(Tx_i) = \delta_{i,k}$.
Since $det(\{\delta_{i,k}\}_{i,k}) = 1$, we have a $\sigma > 0$ such as :
For $\{\beta_{i,k}\}_{i,k}$ with $|\delta_{i,k} - \beta_{i,k}| \leq \sigma$, then $det(\{\beta_{i,k}\}_{i,k}) \neq 0$.
Let's fix $\varepsilon > 0$. Since $\alpha_r(T) = 0$, We have $A \in \mathcal{L}(E, F)$, $rank(A) \leq r$, such as $\|T - A\| \leq \varepsilon$.
Thus, we have in particular : $|\delta_{i,k} - \langle b_k, Ax_i \rangle| = |\langle b_k; Tx_i - Ax_i \rangle| \leq \|T - A\|.\|x_i\|.\|b_k\| \leq \varepsilon \|x_i\|.\|b_k\| \leq \sigma$, for ε small enough.
Therefore, $det(\{\langle b_k; Ax_i \rangle\}_{i,k}) \neq 0 \Rightarrow Ax_i$ are linearly independant $\Rightarrow rank(A) \geq r + 1$, contradiction. □

7.6 Proposition. — *$\forall r, s \geq 0$,*

$$\alpha_{r+s}(ST) \leq \alpha_r(S).\alpha_s(T)$$

Proof. Let $\varepsilon > 0$. We choose $A, B \in \mathcal{L}(E, F)$ with $rank(A) \leq s$, $\|T - A\| \leq \alpha_s(T) + \varepsilon$, $rank(B) \leq r$, $\|S - B\| \leq \alpha_r(S) + \varepsilon$.
Then, $rank(A(T - B) + SB) \leq s + r$, and we have :

$$\alpha_{r+s}(ST) \leq \|ST - A(T - B) - SB\| \leq \|S - A\|.\|T - B\| \leq \alpha_r(S)\alpha_s(T) + \varepsilon^2 + \varepsilon(\alpha_r(S) + \alpha_s(T))$$

□

7.7 Note. *If G is a linear subspace of F, then $\alpha_r^F(T) \leq \alpha_r^G(T)$ as $\mathcal{L}(E, G) \subset \mathcal{L}(E, F)$.*

7.8 Proposition. — *If G is dense in F, then $\alpha_r^F(T) = \alpha_r^G(T)$.*

7.9 Lemma. *Let $r \geq 0$. If we have $T \in \mathcal{L}(E, \mathbb{C}^{r+1})$ such as $\exists S \in \mathcal{L}(\mathbb{C}^{r+1}, E)$ with $TS = I_{\mathbb{C}^{r+1}}$. Then, $\forall 0 \leq n \leq r$,*

$$\alpha_n(T).\|S\| \geq 1$$

Proof. If we have $0 \leq n \leq r$ with $\alpha_n(T).\|S\| < 1$, we have A with $rank(A) \leq n$ such as $\|T - A\|.\|S\| < 1$.
Thus, $I_{\mathbb{C}^{r+1}} - (T - A)S$ must be invertible in $\mathcal{L}(\mathbb{C}^{r+1})$.
But $I_{\mathbb{C}^{r+1}} - (T - A)S = TS - TS + AS = AS$, non invertible because $rank(AS) \leq r$, contradiction. □

7.10 Proposition. •— *For $T \in \mathcal{L}(l^2)$ with $T((x_i)_i) = (\tau_i.x_i)_i$, $\tau_i \in \mathbb{C}$, then*

$$\alpha_r(T) = \sup_{I \subset \mathbb{N},\, card(I) = r+1} \{\inf\{|\tau_i|,\, i \in I\}\} := \sigma_r(T)$$

Proof. We note that for $I_0 := \{i \in \mathbb{N}$ such as $|\tau_i| > \sigma_r(T)\}$, $card\{I_0\} \leq r$.
And $A : (x_i)_i \mapsto (\tau_i.\delta_{I_0}(i)x_i)_i$ is of rank $\leq r$.
Thus, $\alpha_r(T) \leq \|T - A\| = \sup\{|\tau_i|, i \in \mathbb{N} - I_0\} \leq \sigma_r(T)$.
If $\sigma_r(T) = 0$, we have the equality we wanted.
Else, take $I \subset \mathbb{N}$, $card(I) = r + 1$, with $\rho_I := \inf\{|\tau_i|, i \in I\} > 0$. Since $\sigma_r(T) > 0$, such an I that gives $\rho_I > 0$ exists.
Let $P_I : (x_i)_i \mapsto (\delta_I(i)x_i)_i$. We have $rank(P_i) = r + 1$ and $\|P_I\| = 1$.
We have $P_I.T : (x_i)_i \mapsto (\tau_i.\delta_I(i)x_i)_i$.
Thus, for $S_I : (x_i)_i \mapsto (\gamma_i.x_i)_i$ with $\left| \begin{array}{l} \frac{1}{\tau_i} \text{ if } i \in I \\ 0 \text{ else.} \end{array} \right.$
We have $\|S_I\| = \frac{1}{\inf\{|\tau_i|\}} = \frac{1}{\rho_I}$, and $(P_I.T).S_I((x_i)_{i \in I}) = (x_i)_{i \in I}$.

Thus, by lemma 7.9, $\alpha_r(P_I.T).\|S\| \geq 1 \Rightarrow \alpha_r(P_I.T) \geq \rho_I$
$\Rightarrow \rho_I \leq \alpha_r(P_I.T) \leq \|P_I\|\alpha_r(T) \leq \alpha_r(T)$.
Since this is true for all I for which $card(I) = r+1$ and $\rho_I > 0$, and since $\sigma_r(T) = sup\{\rho_J, card(J) = r+1\}$, we have $\sigma_r(T) \leq \alpha_r(T)$. □

7.11 Note. *This last property is also true for l^p with $0 < p \leq \infty$.*
It will be useful in certain cases like for self-adjoint compact operators, as if you use the orthonormal set that diagonalizes the operator, you can see it as a bounded operator on l^2 and obtain two different expressions of $\alpha_r(T)$: one as an infimum, and one as the supremum of an infimum.

We now have all the tools necessary for a good study of approximation numbers. We will now begin to define spaces of bounded operators with specific approximation numbers.

7.2 Operators of type l^p

7.12 Definition. Let $0 < p < \infty$. We define $\mathcal{I}_p(E,F) := \{T \in \mathcal{L}(E,F)$ such as$\sum_r \alpha_r(T)^p < \infty\}$.
These operators are called **type** l^p operators.

7.13 Proposition. — *$\mathcal{I}_p(E,F)$ is a vector space.*

Proof. First, $\mathcal{I}_p(E,F) \subset \mathcal{L}(E,F)$. Let $\lambda \in \mathbb{C}$, $T, S \in \mathcal{I}_p$. We have :
$\alpha_r(\lambda.T) = |\lambda|.\alpha_r(T) \Rightarrow \lambda.T \in \mathcal{I}_p$.
We remind that $\forall a, b \geq 0$, $(a+b)^p \leq \max\{2^{p-1}, 1\}.(a^p + b^p)$. We note $\tau_p := \max\{2^{p-1}, 1\}$.
Thus,

$$\begin{aligned}
\sum_{r\geq 0} \alpha_r(S+T)^p \leq \sum_{r\in 2\mathbb{N}} \alpha_r(S+T)^p + \sum_{r\in(2\mathbb{N}+1)} \alpha_{r-1}(S+T)^p &\leq 2.\sum_{r\geq 0} \alpha_{2r}(S+T)^p \\
&\leq 2.\sum_{r\geq 0} (\alpha_r(S) + \alpha_r(T))^p \\
&\leq 2\tau_p.\sum_{r\geq 0} (\alpha_r(S)^p + \alpha_r(T)^p) < \infty
\end{aligned}$$

□

7.14 Definition. For $0 < p < \infty$, $T \in \mathcal{I}_p$, we define $Q_p(T) := (\sum_{r\geq 0} \alpha_r(T)^p)^{\frac{1}{p}}$.

7.15 Proposition. *$Q_p(T)$ verifies the following properties :*
i) $Q_p(T) \geq 0$
ii) $Q_p(T) = 0 \Leftrightarrow T = 0$.
iii) $\forall \lambda \in \mathbb{C}$, $Q_p(\lambda.T) = |\lambda| Q_p(T)$
iv) For $\rho_p := \left| \begin{array}{l} 2 \text{ if } p \geq 1 \\ 2^{\frac{1}{p-1}} \text{ if } p < 1 \end{array} \right.$, *$\forall S, T \in \mathcal{I}_p$, we have :*
$Q_p(S+T) \leq \rho_p(Q_p(T) + Q_p(S))$.

Thus, Q_p is a quasi-norm on $\mathcal{I}_p$. It is not a norm as it doesn't verify the triangular inequality, but it still defines a metric topology on $\mathcal{I}_p$.
For certain p like $p = 1, 2$, the inequality can be refined to show that Q_p is a norm.

7.16 Proposition. — *Let's suppose here that E=F.*
- For $T \in \mathcal{I}_p$, $Q_p(T) \geq \alpha_0(T) = \|T\|$.
- For $S, T \in \mathcal{I}_p$, $Q_p(S.T) \leq Q_p(S).Q_p(T)$.
- $\mathcal{I}_p$ is an ideal : $\forall T \in \mathcal{I}_p$, $\forall A \in \mathcal{L}(E)$, $AT, TA \in \mathcal{I}_p$.
$\alpha_r(AT) \leq \alpha_r(T).\|A\| \Rightarrow Q_p(AT) \leq Q_p(T).\|A\|$.
$\alpha_r(TA) \leq \alpha_r(T).\|A\| \Rightarrow Q_p(TA) \leq Q_p(T).\|A\|$.
- If E is an Hilbert space, for $T \in \mathcal{I}_p$, then $T^* \in \mathcal{I}_p$.
Since $rank(A) = rank(A^*)$ and $\|T^* - A^*\| = \|T - A\|$, $\alpha_r(T) = \inf\{\|T - A\|, rank(A) \leq r\} = \inf\{\|T^* - A\|, rank(A) \leq r\} = \alpha_r(T^*)$.

7.17 Lemma. *If $\{T_n\}_n$ is Q_p-Cauchy in $\mathcal{I}_p$, then it converges towards a $T \in \mathcal{I}_p$ for Q_p.*
Thus, $\mathcal{I}_p(E, F)$ is complete with Q_p.

Proof. Since $Q_p(.) \geq \|.\|$, $\{T_n\}_n$ is also $\|.\|$-Cauchy. Thus, it converges towards a $T \in \mathcal{L}(E, F)$ for $\|.\|$.
We have : $|\alpha_r(T - T_n) - \alpha_r(T_m - T_n)| \leq \|T - T_n - T_m + T_n\| = \|T - T_m\|$.
Thus, $\forall n \geq 0$, $\alpha_r(T_m - T_n) \to_{m\to\infty} \alpha_r(T - Tn)$
Let $\varepsilon > 0$. $\exists n_0$ such as $\forall m, q \geq n_0$, $Q_p(T_m - T_q) = (\sum_{r\geq 0} \alpha_r(T_m - T_q)^p)^{\frac{1}{p}} < \varepsilon$.

$$\Rightarrow \; Q_p(T - T_q) = (\sum_{r\geq 0} \alpha_r(T - T_q)^p)^{\frac{1}{p}} < \varepsilon$$

by uniform dominated convergence of $\sum_{r=0}^{N} \alpha_r(T_m - T_q)^p$ towards $\sum_{r=0}^{N} \alpha_r(T - T_q)^p$, $\forall N \geq 0$.
Thus, $T - T_q \in \mathcal{I}_p \Rightarrow T \in \mathcal{I}_p$, and $T_n \xrightarrow{Q_p}_{n\to\infty} T$. □

7.18 Proposition. — *The finite rank operators are dense in $\mathcal{I}_p$.*

Proof. Let $T \in \mathcal{I}_p$. Let $\varepsilon > 0$. $\exists n_0$ such as $\sum_{r\geq n_0} \alpha_r(T)^p < \varepsilon$.
Thus, $n_0.\alpha_{2n_0}(T)^p \leq \sum_{r=n_0+1}^{2n_0} \alpha_r(T)^p < \varepsilon$.
Let's take $A \in \mathcal{L}(E, F)$, $rank(A) \leq 2n_0$, with $n_0.\|T - A\|^p < \varepsilon$.
Then, $\alpha_{r+2n_0}(T - A) \leq \alpha_r(T) + \alpha_{2n_0}(A) = \alpha_r(T) + 0$, $\forall r \geq 0$.
Therefore :

$$\begin{aligned} Q_p(T - A)^p &= \sum_{r=0}^{3n_0-1} \alpha_r(T - A)^p + \sum_{r=3n_0}^{\infty} \alpha_r(T - A)^p \\ &\leq 3n_0.\|T - A\|^p + \sum_{r=n_0}^{\infty} \alpha_r(T)^p \\ &\leq 3\varepsilon + \varepsilon \end{aligned}$$

Thus, for every $T \in \mathcal{I}_p$, for every $\varepsilon > 0$, we have a finite rank operator A such as $Q_p(T - A) < \varepsilon$. □

7.19 Proposition. — *Since we know that $\overline{\{\text{finite rank}\}}^{Q_p} = \mathcal{I}_p$ and that $Q_p(.) \geq \|.\|$, we have :*

$$\mathcal{I}_p = \overline{\{\text{finite rank}\}}^{Q_p} \subset \overline{\{\text{finite rank}\}}^{\|.\|} = K(E, F)$$

So every operator in $\mathcal{I}_p$ is compact.

7.20 Proposition. — *For $p \leq q$, $\mathcal{I}_p \subset \mathcal{I}_q$.*

Proof. If $Q_p(T) < \infty$, then $\exists n_0$ such as $\forall n \geq n_0$, $0 \leq \alpha_n(T) < 1$.
$\Rightarrow \alpha_n(T)^q \leq \alpha_n(T)^p \Rightarrow \sum_{n=n_0}^{\infty} \alpha_n(T)^q < \infty$. □

7.21 Theorem. *$\forall T \in \mathcal{I}_p(E,F)$, $\forall S \in \mathcal{I}_q(F,G)$, we have $ST \in \mathcal{I}_s(E,G)$, with $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$, and $Q_s(ST) \leq 2^{\frac{1}{s}} Q_q(S) Q_p(T)$.*

Proof. We have :

$$\begin{aligned} Q_s(ST) = (\sum_{r\geq 0} \alpha_r(ST)^s)^{\frac{1}{s}} \leq (2.\sum_{r\geq 0} \alpha_{2r}(ST)^s)^{\frac{1}{s}} &\leq (2.\sum_{r\geq 0} \alpha_r(S)^s.\alpha_r(T)^s)^{\frac{1}{s}} \\ &\leq 2^{\frac{1}{s}}.(\sum_{r\geq 0} \alpha_r(S)^q)^{\frac{1}{q}}.(\sum_{r\geq 0} \alpha_r(T)^p)^{\frac{1}{p}}, \text{ using Holder's inequality} \\ &\leq 2^{\frac{1}{s}} Q_q(S) Q_p(T) < \infty \end{aligned}$$

□

7.22 Note. *For $T \in \mathcal{I}_p$, $\sum_r \alpha_r(T)^p < \infty$.*
Thus, $\forall N \geq 0$, $(N+1).\alpha_N(T)^p \leq \sum_{n=0}^N \alpha_n(T)^p \leq Q_p(T)^p$
$\Rightarrow \alpha_N(T) \leq \frac{Q_p(T)}{(N+1)^{\frac{1}{p}}}$.

7.23 Proposition. *We have then :*

$$\alpha_n(T) \leq \frac{C}{(N+1)^{\frac{1}{p}+\varepsilon}} \Rightarrow T \in \mathcal{I}_p \Rightarrow \alpha_n(T) \leq \frac{\widetilde{C}}{(N+1)^{\frac{1}{p}}}$$

7.24 Proposition. *— For $T \in \mathcal{L}(l^2)$ with $T((x_n)_n) = (\tau_n x_n)_n$, $Q_p(T)^p = \sum_i |\tau_i|^p$.*
So $T \in \mathcal{I}_p \Leftrightarrow (\sum_i |\tau_i|^p) < \infty$.

Proof. From proposition 7.10, we have $\alpha_r(T) = \sup_{I\subset\mathbb{N},\, card(I)=r+1}\{\inf\{|\tau_i|,\, i \in I\}\}$.
Thus, $T \in \mathcal{I}_p \Rightarrow \alpha_r(T) \to_{r\to\infty} 0$.
This implies that $\forall n \geq 1$, $card\{i \in \mathbb{N}$ such as $|\tau_i| \geq \frac{1}{n}\} < \infty$, else the sequence of $\alpha_r(T)$ can't converge towards 0.
And if $\sum_i |\tau_i|^p < \infty$, $\forall n \geq 1$, $card\{i \in \mathbb{N}$ such as $|\tau_i| \geq \frac{1}{n}\} < \infty$ too.
Thus, in both cases, we can reorder the τ_i by decreasing modulus with a permutation σ, as there is always a finite amount of τ_i of modulus higher than $\frac{1}{n}$.
We end up with $(\tau_{\sigma(i)})_i$, such as $|\tau_{\sigma(0)}| \geq |\tau_{\sigma(1)}| \geq |\tau_{\sigma(2)}| \geq \ldots \geq 0$.
Thus, $\alpha_r(T) = |\tau_{\sigma(r)}|$ and $Q_p(T)^p = \sum_{i\geq 0} |\tau_{\sigma(i)}|^p = \sum_{i\geq 0} |\tau_i|^p$. □

7.25 Note. *Similarly to proposition 7.10, this proposition is also true for l^p with $0 < p \leq \infty$. Since this proposition will be useful when we will look at Hilbert spaces, l^2 seemed more appropriate here.*

7.26 Proposition. *— $\forall p \geq q \geq 0$, $\forall T \in \mathcal{I}_q$, T is in $\mathcal{I}_p$ and $Q_p(T) \leq Q_q(T)$.*

To prove this proposition, we will need to use a lemma on series of real positive numbers.

7.27 Lemma. *For $(\gamma_n)_n \in \mathbb{R}_+^{\mathbb{N}}$ with $\sum_n \gamma_n < \infty$, $\forall \beta \geq 1$, we have :*

$$(\sum_{n\geq 0} \gamma_n^\beta) \leq (\sum_{n\geq 0} \gamma_n)^\beta$$

Proof. Since $\sum_{n\geq 0} \gamma_n < \infty$, $(\sum_{n\geq 0} \gamma_n^\beta)$ too because $\exists n_0$ such as $\forall n \geq n_0$, $\gamma_n < 1 \Rightarrow \gamma_n^\beta \leq \gamma_n$.
We will look at $(\sum_{n\geq 0} \gamma_n^\beta) - (\sum_{n\geq 0} \gamma_n)^\beta$.
For $n \geq 1$, we define $f_n : (x_0, \ldots, x_n) \in \mathbb{R}_+^{n+1} \mapsto (x_o + \ldots + x_n)^\beta - (x_o^\beta + \ldots + x_n^\beta) \in \mathbb{R}$.
f_n is of C^∞-class on $\mathbb{R}_+^{n+1}$.

We have $\partial_{x_i} f_n(x_o, \ldots, x_n) = \beta.(x_o + \ldots + x_n)^{\beta-1} - \beta.x_i^{\beta-1} \geq 0$, $\forall x_0, \ldots, x_n \geq 0$.
Thus, $f_n(x_0, \ldots, x_n) \geq f_n(x_0, \ldots, x_{n-1}, 0) \geq \ldots \geq f_n(0, \ldots, 0) = 0$.
Therefore, $0 \leq (\gamma_o + \ldots + \gamma_n)^\beta - (\gamma_o^\beta + \ldots + \gamma_n^\beta) \leq (\sum_{n\geq 0} \gamma_n^\beta) + (\sum_{n\geq 0} \gamma_n)^\beta := M$, $\forall n \geq 0$.
Since $\gamma_0 + \ldots + \gamma_n \to \sum_n \gamma_n$ and $\gamma_0^\beta + \ldots + \gamma_n^\beta \to \sum_n \gamma_n^\beta$, we end up with :

$$0 \leq (\sum_{n\geq 0} \gamma_n^\beta) - (\sum_{n\geq 0} \gamma_n)^\beta \leq M$$

by dominated convergence. □

Proof. We have $Q_p(T) = (\sum_r \alpha_r(T)^p)^{\frac{1}{p}}$ and $Q_q(T) = (\sum_r \alpha_r(T)^q)^{\frac{1}{q}}$.
We note $\gamma_n = \alpha_n(T)^q \Rightarrow \alpha_n(T)^p = \gamma_n^{\frac{p}{q}}$, and $\frac{p}{q} \geq 1$.
Thus, $Q_p(T) \leq Q_q(T) \Leftrightarrow (\sum_n \gamma_n^{\frac{p}{q}}) \leq (\sum_n \gamma_n)^{\frac{p}{q}}$, which is true thanks to lemma 7.27. □

7.28 Note. *All the Q_p are now ordered. For $T \in \mathcal{L}(E, F)$, we have :*

$$\|T\| \leq \ldots \leq Q_2(T) \leq \ldots \leq Q_1(T) \leq \ldots \leq \lim_{p\to 0^+} (Q_p(T)) \leq \infty$$

7.3 Operators of type s

7.29 Definition. We define $s(E, F) := \cap_{p>0} \mathcal{I}_p(E, F)$.
An operator T in s is called a **type s** operator.

Since $\mathcal{I}_p$ are all ideals of $\mathcal{L}(E, F)$, s is an ideal of $\mathcal{L}(E, F)$.
Finite rank operators are all of type s.
In the case of E=F, E Hilbert, s is a *-ideal. $T \in s \Rightarrow T^* \in s$.
T is of type s $\Leftrightarrow \sum_{r\geq 0} \alpha_r(T)^p < \infty, \forall 0 < p < \infty$.

7.30 Definition. We define a metric topology on $s(E, F)$ with :
$U_{p,\varepsilon}(T) := \{S \in s(E, F)$ such as $Q_p(T - S) \leq \varepsilon\}$ as a fundamental system of neighborhoods of T.

7.31 Proposition. — *This topology is well defined as a metric topology.*
With this topology, the finite rank operators are dense in s and s is complete.

7.32 Note. *We have :*

$$\{\textit{finite rank}\} \subset s \subset \ldots \subset \mathcal{I}_1 \subset \ldots \subset \mathcal{I}_2 \subset \ldots \subset K$$

7.33 Note. *For $T \in s$, $\sum_r \alpha_r(T)^p < \infty$, $\forall 0 < p < \infty$.*
Thus, $\forall N \geq 0$, $\forall 0 < p < \infty$, $(N + 1).\alpha_N(T)^p \leq \sum_{n=0}^{N} \alpha_n(T)^p \leq Q_p(T)^p$
$\Rightarrow \alpha_N(T) \leq \frac{Q_p(T)}{(N+1)^{\frac{1}{p}}}$, $\forall 0 < p < \infty$.

7.34 Proposition. — *$T \in s(E, F) \Leftrightarrow (\alpha_n(T))_n$ is a rapidly decreasing sequence :*
$\forall p > 0$, $\exists C > 0$ such as $\alpha_n(T) \leq \frac{C}{(N+1)^{\frac{1}{p}}}$

7.4 Approximation numbers of compact mappings in Hilbert spaces

We will now suppose that E and F are separable complex Hilbert spaces.

7.35 Theorem. *Let $T \in \mathcal{L}(E,F)$.*
- T is compact $\Leftrightarrow \alpha_r(T) \to_{r\to\infty} 0$.
- If T is compact, for $(\lambda_n)_n$ the singular values of T, we have :

$$\lambda_n = \alpha_n(T),\ \forall n \geq 0$$

- If T is compact, then

$$T \in \mathcal{I}_p \ \Leftrightarrow\ \sum_{n\geq 0} \lambda_n^p < \infty$$

Proof.
$\Leftarrow$
If $\alpha_r(T) \to 0$, then we have a sequence of finite rank operators $(A_n)_n$ such as $\|T - A_n\| \leq 2\alpha_n(T) \Rightarrow \|T - A_n\| \to 0 \Rightarrow (A_n)_n$ converges towards T in operator norm.
$\Rightarrow$
If T is compact, let $\{\phi_n\}_n$,$\{\psi_n\}_n$ be orthonormal sets of E and F for which :
$T(f) = \sum_n \lambda_n.\langle\phi_n; f\rangle_E.\psi_n$, with $(\lambda_n)_n \in \mathbb{R}_+^{\mathbb{N}}$ monotone decreasing towards 0.
By writing $x \in E$ as $x = \sum_n x_n.\phi_n + x^\perp$, and $y \in F$ as $y = \sum_n y_n.\psi_n + y^\perp$ with $(x_n)_n, (y_n)_n \in l^2$, we can see T as :
$T : ((x_n)_n, x^\perp)_E \mapsto ((\lambda_n.x_n)_n, 0)_F$
Thus, by using arguments from 7.24, we obtain :

$$\forall r \geq 0,\ \alpha_r(T) = \sup_{I\subset\mathbb{N},\ card(I)=r+1} \{\inf\{\lambda_i,\ i \in I\}\} = \lambda_r$$

Which completes the proof. □

This theorem links approximation numbers with singular values. It will be useful to give more properties to two specific $\mathcal{I}_p$, since it gives new ways to express approximation numbers through the Hilbert-Schmidt theorem.

7.5 Trace-class ideal

Let H be a separable complex Hilbert space.

7.36 Definition. Let $T \in \mathcal{L}(H)$ a positive operator. Let $\{\psi_n\}_n$ be an orthonormal basis of E.
We define $Tr(T) := \sum_n \langle\psi_n; T\psi_n\rangle \in [0,\infty]$, the **trace** of T.

7.37 Proposition. — *Let $T \in \mathcal{L}(H)$ a positive operator. Then $Tr(T)$ is independant of the orthonormal basis chosen.*
For S another positive operator, $Tr(S+T) = Tr(S) + Tr(T)$.
For $\lambda \geq 0$, $Tr(\lambda T) = \lambda.Tr(T)$.
If U is an unitary operator, then $Tr(UTU^{-1}) = Tr(T)$.

Proof. Let $\{\phi_n\}_n$ be another orthonormal basis of H. Then :

$$\begin{aligned}
Tr_\psi(T) = \sum_{n\geq 0}\langle\psi_n; T\psi_n\rangle = \sum_{n\geq 0}\|\sqrt{T}\psi_n\|^2 &= \sum_{n\geq 0}(\sum_{m\geq 0}|\langle\phi_m; \sqrt{T}\psi_n\rangle|^2) \\
&= \sum_{m\geq 0}(\sum_{n\geq 0}|\langle\sqrt{T}\phi_m; \psi_n\rangle|^2) \text{ because } \sqrt{T} \text{ is self-adjoint} \\
&= \sum_{m\geq 0}\|\sqrt{T}\phi_m\|^2 \\
&= \sum_{m\geq 0}\langle\phi_m; T\phi_m\rangle = Tr_\phi(T)
\end{aligned}$$

The linearity of the trace is obvious.
And for U an unitary operator, $\{U^{-1}\psi_n\}_n$ is another orthonormal basis of H, so $\sum_{n\geq 0}\langle \psi_n; T\psi_n\rangle = \sum_{n\geq 0}\langle U^{-1}\psi_n; TU^{-1}\psi_n\rangle$.

□

7.38 Definition. An operator $T \in \mathcal{L}(H)$ is called **Trace-class** if and only if $Tr(|T|) < \infty$.

7.39 Proposition. *If A, B are trace-class, then $A+B$ is trace-class and $Tr(|A+B|) \leq Tr(|A|)+Tr(|B|)$*

Proof. The polar decomposition theorem gives us U,V,W partial isometries for which :
$A + B = U|A + B|$ and $Ker(U) = Ker(A + B) = Ker(|A + B|)$,
$A = V|A|$ and $Ker(V) = Ker(A) = Ker(|A|)$,
$B = W|B|$ and $Ker(W) = Ker(B) = Ker(|B|)$.
Let $\{\psi_n\}_n$ be an orthonormal basis of H. Let $N \geq 0$.
Then,

$$\sum_{n=0}^{N}\langle \psi_n, |A+B|\psi_n\rangle = \sum_{n=0}^{N}\langle \psi_n, U^*(A+B)\psi_n\rangle \leq \sum_{n=0}^{N}|\langle \psi_n, U^*V|A|\psi_n\rangle| + \sum_{n=0}^{N}|\langle \psi_n, U^*W|B|\psi_n\rangle|$$

And,

$$\begin{aligned}\sum_{n=0}^{N}\langle \psi_n, U^*V|A|\psi_n\rangle &\leq \sum_{n=0}^{N}\|\sqrt{|A|}V^*U\psi_n\|.\|\sqrt{|A|}\psi_n\| \\ &\leq (\sum_{n=0}^{N}\|\sqrt{|A|}V^*U\psi_n\|^2)^{\frac{1}{2}}.(\sum_{n=0}^{N}\|\sqrt{|A|}\psi_n\|^2)^{\frac{1}{2}} \\ &\leq Tr(|A|)^{\frac{1}{2}}.Tr(|A|)^{\frac{1}{2}} \text{ as } \{V^*U\psi_n\}_n = \{\delta_n\phi_n\},\ \phi_n \text{ an ONB and } \delta_n = \left|\begin{array}{l} 0 \text{ if } \psi_n \in Ker(V^*U) \\ 1 \text{ else.}\end{array}\right. \\ &\leq Tr(|A|)\end{aligned}$$

Thus, $Tr(|A + B|) \leq Tr(|A|) + Tr(|B|) < \infty$. □

7.40 Proposition. *If T is trace-class, then T is compact.*
Furthermore, T is in $\mathcal{I}_1$ and $Tr(|T|) = Q_1(T)$.
Therefore, $\forall T \in \mathcal{L}(H)$, $Tr(T) = Q_1(T)$, and the trace-class operators are exactly the operators in $\mathcal{I}_1$.

Proof. Since T is trace-class, $Tr(|T|) = \sum_{n\geq 0}\|\sqrt{|T|}\psi_n\|^2 < \infty$.
Thus, $f \in H \mapsto \sum_{n=0}^{N}\langle \psi_n; \sqrt{|T|}f\rangle.\psi_n$ is of finite rank and converges in operator norm towards $\sqrt{|T|}$:

$$\|\sum_{n=0}^{N}\langle \psi_n; \sqrt{|T|}.\rangle\psi_n - \sqrt{|T|}\| \leq \sum_{n=N+1}^{\infty}\|\sqrt{|T|}\psi_n\|^2 \to_{N\to\infty} 0$$

Thus, $\sqrt{|T|}$ is compact, so $|T|$ and T are compact.
The Hilbert-Schmidt theorem gives us an orthonormal set $\{\phi_n\}_n$ and $(\lambda_n)_n \in \mathbb{R}_+^{\mathbb{N}}$ monotone decreasing towards 0 such as : $|T| = \sum_n \lambda_n\langle \phi_n; .\rangle_E\phi_n$.
We complete $\{\phi_n\}_n$ by another orthonormal set $\{\Phi_n\}_n$ to obtain an orthonormal basis of H. Every Φ_n is in particular in $Ker(|T|)$.
Thus,

$$Tr(|T|) = \sum_{n\geq 0}\langle \phi_n; T\phi_n\rangle + \sum_{n\geq 0}\langle \Phi_n; T\Phi_n\rangle = \sum_{n\geq 0}\lambda_n + 0$$

As the $(\lambda_n)_n$ are the singular values of T, $Q_1(T) = \sum_{n\geq 0} \alpha_n(T) = \sum_{n\geq 0} \lambda_n = Tr(|T|) < \infty$, so T is in $\mathcal{I}_1(H)$.
Conversely, if T is in $\mathcal{I}_1(H)$, then T and $|T|$ are compact. A second use of the Hilbert-Schmidt theorem gives us : $Tr(|T|) = \sum_{n\geq 0} \lambda_n = Q_1(T) < \infty$.
So T is trace-class, and $Tr(|T|) = Q_1(T)$ for any T in $\mathcal{L}(H)$. □

With this proposition, all the properties found for $Q_1(T)$ transfer to $Tr(|T|)$, and vice-versa. We have in particular :

7.41 Proposition. — *As Q_1 is a quasi-norm and $Q_1(T+S) = Tr(|T+S|) \leq Tr(|T|) + Tr(|S|) = Q_1(T) + Q_1(S)$, Q_1 is a norm on $\mathcal{I}_1(H)$.*
Thus, $(\mathcal{I}_1(H), Q_1)$ is a Banach space.

By knowing an orthonormal basis of H, and if T is positive or if $|T|$ is easily computable, we can try to compute $Tr(|T|)$ in order to see if T is in $\mathcal{I}_1$ or not.
This can be used to prove that T is compact as proving that T is in a certain $\mathcal{I}_p$ is sometimes easier.

7.42 Proposition. — *Let $T \in \mathcal{I}_1(H)$. Let $\{\psi_n\}_n$ be an orthonormal basis of H.*
Then, $Tr(T) := \sum_{n\geq 0} \langle \psi_n ; T\psi_n \rangle$ exists, and doesn't depend on the orthonormal basis chosen.
We have in particular that $|Tr(T)| \leq Tr(|T|)$.

Proof. Let $N \geq 0$. We have U a partial isometry such as $T = U|T|$. This gives us :

$$\begin{aligned}
|\sum_{n=0}^{N} \langle \psi_n, T\psi_n \rangle| \leq \sum_{n=0}^{N} |\langle \psi_n, U|T|\psi_n \rangle| &\leq \sum_{n=0}^{N} \|\sqrt{|T|}U^*\psi_n\| . \|\sqrt{|T|}\psi_n\| \\
&\leq (\sum_{n=0}^{N} \|\sqrt{|T|}U^*\psi_n\|^2)^{\frac{1}{2}} . (\sum_{n=0}^{N} \|\sqrt{|T|}\psi_n\|^2)^{\frac{1}{2}} \\
&\leq Tr(|T|)^{\frac{1}{2}} . Tr(|T|)^{\frac{1}{2}} \text{with the same argument as in} \\
&\leq Tr(|T|) < \infty
\end{aligned}$$

The proof of the independance of the orthonormal basis is the same as the one for $Tr(|T|)$. □

With the trace-class ideal properly defined and studied, we can now introduce the Hilbert-Schmidt ideal. The study of approximation numbers is already of a great help here.

7.6 Hilbert-Schmidt ideal

7.43 Definition. An operator T $\in \mathcal{L}(H)$ is called **Hilbert-Schmidt** if and only if $Tr(TT^*) < \infty$.

7.44 Proposition. — *Let $\{\psi_n\}_n$ be an orthonormal basis of H. We have :*

$$Tr(TT^*) = \sum_{n\geq 0} \langle \psi_n, TT^*\psi_n \rangle = \sum_{n\geq 0} \|T\psi_n\|^2$$

Thus, if T is an Hilbert-Schmidt operator, T is compact, T is in $\mathcal{I}_2(H)$, and :

$$Q_2(T) = \sqrt{Tr(TT^*)} = \sqrt{Q_1(TT^*)}$$

As the converse is true, the Hilbert-Schmidt operators are exactly the operators in $\mathcal{I}_2(H)$.
We also have that :

$$\begin{aligned} Q_2(A+B)^2 = Tr((A+B)(A+B)^*) = \sum_{n\geq 0} \|(A+B)\psi_n\|^2 &\leq \sum_{n\geq 0} \|A\psi_n\|^2 + \sum_{n\geq 0} \|T\psi_n\|^2 \\ &\leq Tr(AA^*) + Tr(BB^*) \\ &\leq Q_2(A)^2 + Q_2(B)^2 \end{aligned}$$

Thus, the quasi-norm Q_2 is in fact a norm in $\mathcal{I}_2(H)$

7.45 Note. *$\langle T, S\rangle_{\mathcal{I}_2} = Q_1(TS^*)$ is an inner product on $\mathcal{I}_2(H)$.*

7.46 Theorem. — *Let $A \in \mathcal{L}(H)$.*
$A \in \mathcal{I}_1(H) \Leftrightarrow A = BC$ with $B, C \in \mathcal{I}_2(H)$.

Proof.
$\Leftarrow$
If $A = BC$ with $B, C \in \mathcal{I}_2(H)$, since $\frac{1}{2} + \frac{1}{2} = \frac{1}{1}$, then A is in $\mathcal{I}_1(H)$.
$\Rightarrow$
If A is in $\mathcal{I}_1$, for λ_n the singular values of A, $\sqrt{\lambda_n}$ are the singular values of $\sqrt{|A|}$.
Thus, $Q_2(\sqrt{|A|})^2 = \sum_{n\geq 0} \lambda_n = Q_1(A) < \infty$, so $\sqrt{|A|} \in \mathcal{I}_\in$.
The polar decomposition theorem gives us a partial isometry U such as $A = U.|A| = U\sqrt{|A|}.\sqrt{|A|}$. By choosing $B = U\sqrt{|A|}$ and $C = \sqrt{|A|}$, we complete the proof. □

In certain Hilbert spaces, belonging to the $\mathcal{I}_2(H)$ can be tested with criterias that do not involve the calculation of the singular values or of Q_2. These criterias are really useful to prove the compacness of certain operators, or their belonging to $\mathcal{I}_1$ or s, especially in RKHS like Bergman and Hardy spaces.

7.47 Theorem. — *Let (M,μ) be a mesured space and $H = L^2(M, d\mu)$.*
Then, $A \in \mathcal{L}(H)$ is Hilbert-Schmidt $\Leftrightarrow \exists K \in L^2(M\times M, d\mu\otimes d\mu)$ such as $(Af)(x) = \int K(x,y)f(y)d\mu(y)$, $\forall f \in L^2(M, d\mu)$.
We also have : $\|A\|^2 = \iint |K(x,y)|^2 d\mu(x)d\mu(y)$.

7.48 Corollary.
$A \in \mathcal{L}(H)$ is trace-class $\Leftrightarrow \exists K_1, K_2 \in L^2(M\times M, d\mu\otimes d\mu)$ such as $(Af)(x) = \int(\int K_1(x,z)K_2(z,y)d\mu(z))f(y)d\mu(y)$, $\forall f \in L^2(M, d\mu)$.

With the study of Hilbert spaces, approximation numbers, and $\mathcal{I}_p$ ideals, we can now go back to Hardy and Bergman spaces and study multiple properties of certain operators, especially multiplication and composition operators.

8 Various compact operators in B^2 and H^2 spaces

8.1 Hilbert-Schmidt test for B^2 and H^2

8.1 Theorem. *Hilbert-Schmidt test for composition operators Let $\varphi : \mathbb{D} \to \mathbb{D}$, holomorphic. φ can be extended as a continuous function from $\overline{\mathbb{D}}$ to $\overline{\mathbb{D}}$. We have the following criteria :*

$$C_\varphi \in \mathcal{I}_2(H^2(\mathbb{D})) \;\Leftrightarrow\; \int_{\partial\mathbb{D}} K_{H^2(\mathbb{D})}(\varphi(z), \varphi(z))|dz| = \int_0^{2\pi} \frac{1}{1-|\varphi(e^{it})|^2}dt < \infty$$

$$C_\varphi \in \mathcal{I}_2(B^2(\mathbb{D})) \;\Leftrightarrow\; \iint_{\mathbb{D}} K_{B^2(\mathbb{D})}(\varphi(z), \varphi(z))|dz| = \int_0^1 \int_0^{2\pi} \frac{1}{(1-|\varphi(re^{it})|^2)^2} rdrdt < \infty$$

Proof.
We will do the proof for the Hardy case. The proof for the Bergman case is completely similar.
Let $\{\psi_n\}_n$ be an orthonormal basis of $H^2(\mathbb{D})$. These functions are holomorphic on $\mathbb{D}$ and can be extended as L^2 functions on $\overline{\mathbb{D}}$.
We have :

$$
\begin{aligned}
C_\varphi \in \mathcal{I}_2(H^2(\mathbb{D})) \Leftrightarrow Q_2(C_p) < \infty &\Leftrightarrow \sum_{n\geq 0} \|C_\varphi \psi_n\|^2 < \infty \\
&\Leftrightarrow \sum_{n\geq 0} \int_0^{2\pi} \overline{\psi_n(\varphi(e^{it}))}.\psi_n(\varphi(e^{it}))dt < \infty \\
&\Leftrightarrow \int_0^{2\pi} \sum_{n\geq 0} \overline{\psi_n(\varphi(e^{it}))}.\psi_n(\varphi(e^{it}))dt < \infty \\
&\Leftrightarrow \int_0^{2\pi} K_{H^2(\mathbb{D})}(\varphi(e^{it}), \varphi(e^{it}))dt = \int_0^{2\pi} \frac{1}{1-|\varphi(e^{it})|^2} dt < \infty
\end{aligned}
$$

$\square$

This criteria will be useful in many cases as it gives us properties on C_φ when checking the integrability of $\frac{1}{1-|\phi(z)|^2}$ on certain domains.

8.2 Note. *If we also take a multiplication operator M_w with $w \in Hol(\mathbb{D}) \cap L^\infty(\mathbb{D})$, we can generalize the property to weighted composition operators $M_w \circ C_\varphi$:*

$$M_w \circ C_\varphi \in \mathcal{I}_2(H^2(\mathbb{D})) \;\Leftrightarrow\; \int_{\partial\mathbb{D}} |w(z)|^2.K_{H^2(\mathbb{D})}(\varphi(e^{it}), \varphi(e^{it}))|dz| < \infty$$

$$M_w \circ C_\varphi \in \mathcal{I}_2(B^2(\mathbb{D})) \;\Leftrightarrow\; \iint_{\mathbb{D}} |w(z)|^2.K_{B^2(\mathbb{D})}(\varphi(z), \varphi(z))|dz| < \infty$$

8.3 Note. *For Ω an open simply connected space, $\varphi : \Omega \to \Omega$, holomorphic, $\phi : \Omega \to \mathbb{D}$ a biholomorphism, we have $K_{H^2(\Omega)}(z;z) = |\phi'(z)|.K_{H^2(\mathbb{D})}(\phi(z);\phi(z))$ and $K_{B^2(\Omega)}(z;z) = |\phi'(z)|^2.K_{B^2(\mathbb{D})}(\phi(z);\phi(z))$*
- Thus, we can generalize the property to any Hardy or Bergman space :

$$C_\varphi \in \mathcal{I}_2(H^2(\Omega)) \;\Leftrightarrow\; \int_{\partial\Omega} K_{H^2(\Omega)}(\varphi(z), \varphi(z))|dz| < \infty$$

$$C_\varphi \in \mathcal{I}_2(B^2(\Omega)) \;\Leftrightarrow\; \iint_{\Omega} K_{B^2(\Omega)}(\varphi(z), \varphi(z))|dz| < \infty$$

- On spaces Ω where a biholomorphism ϕ can be computed, we can compute $K_{H^2(\Omega)}$ and apply the Hilbert-Schmidt test to composition operators or weighted composition operators on these spaces that have a simple expression.

8.4 Proposition. *Compactness of multiplication operators*
A non-zero multiplication operator M_w is never compact in B^2,H^2.

Proof. If w is non constant, $\overline{Im(w)}$ is not discrete. If $w \equiv \lambda$, then λ is an eigenvalue of infinite multiplicity for M_w. $\square$

8.5 Proposition. — *For $\Omega_1 \subset \Omega_2$ with $|\Omega_1| < \infty$ and $d(\Omega_1, \Omega_2^C) > 0$, then the restriction operator* $\begin{array}{rccc} I: & B^2(\Omega_2) & \to & B^2(\Omega_1) \\ & f & \mapsto & f|_{\Omega_1} \end{array}$ *is of type s.*
A similar version works in Hardy spaces with $|\partial\Omega_1|$ instead of $|\Omega_1|$.

Proof. For $f \in B^2(\Omega_2)$, since $\iint_{\Omega_1} |f(x+iy)|^2 dxdy \leq \iint_{\Omega_2} |f(x+iy)|^2 dxdy$, I is a bounded operator and $\|I\| \leq 1$. Let $\{\psi_n\}_n$ be an orthonormal basis of $B^2(\Omega_2)$.
- We will first prove that I is in $\mathcal{I}_2$.
We have : $\sum_{n\geq 0} \|I\psi_n\|^2_{B^2(\Omega_1)} = \sum_{n\geq 0}(\iint_{\Omega_1} |\psi_n(x+iy)|^2 dxdy) \leq \sum_{n\geq 0} |\Omega_1|.\sup_{z\in\Omega_1}(|\psi_n(z)|^2)$.
And $\sum_n |\psi_n(z)|^2 = \sum_n |\langle k_z, \psi_n\rangle|^2 = \|k_z\|^2 \leq \frac{C}{d(z,\Omega_2^C)} \leq \frac{C}{d(\Omega_1,\Omega_2^C)}$, with C > 0 a constant.
Thus, $\sum_{n\geq 0} \|I\psi_n\|^2 \leq \sup_{z\in\Omega_1}(\sum_n |\Omega_1||\psi_n(z)|^2) \leq |\Omega_1|.\frac{C}{d(\Omega_1,\Omega_2^C)} < \infty$, and I is in $\mathcal{I}_2$.

- Now, for any n > 0 we can build $\Omega_1 = \Omega_{n,0} \subset \ldots \subset \Omega_{n,n} = \Omega_2$, such as :
$|\Omega_{n,i}| < \infty$ and $d(\Omega_{n,i}, \Omega^C_{n,i+1}) > 0$, $\forall 0 \leq i \leq n-1$.
By defining $\begin{array}{rccc} I_{n,i}: & B^2(\Omega_{n,i+1}) & \to & B^2(\Omega_{n,i}) \\ & f & \mapsto & f|_{\Omega_{n,i}} \end{array}$, each $I_{n,i}$ is in $\mathcal{I}_2$ and $I = I_{n,0}.I_{n,1}\ldots In, n-1$.
Thus, $I \in \mathcal{I}_{\frac{2}{n}}$, $\forall n > 0 \Rightarrow$ I is of type s. □

8.2 Compactness criteria in H^2 and B^2

8.6 Theorem. *Let $\varphi : \mathbb{D} \to \mathbb{D}$, holomorphic.*
Then C_φ is compact on $H^2(\mathbb{D}) \Leftrightarrow \forall (f_n)_n \in H^2(\mathbb{D})^{\mathbb{N}}$ such as $\begin{array}{l} \|f_n\|_{H^2} \textit{ is uniformly bounded by a constant } C > 0 \\ f_n \to 0 \textit{ uniformly on every compact} \end{array}$
$\Rightarrow \|C_\varphi(f_n)\|_{H^2} \to 0$.
We also have the same equivalence for $B^2(\mathbb{D})$.

Proof. We will do the proof for the Hardy space case. The proof for the Bergman space case works the same way.
$\Longleftarrow$ Let $(f_n)_n \in H^2(\mathbb{D})^{\mathbb{N}}$ with $\|f_n\|_{H^2} \leq 1$.
Then, $(f_n)_n$ is uniformly bounded on every compact.
Thus, by a diagonal process on the $\overline{B(0, 1-\frac{1}{m})}$, we have a subsequence $(f_{n_k})_k$ that converges uniformly on every compact towards f that is holomorphic.
We must show that $f \in H^2$.
Let $\varepsilon > 0$ and $0 < r < 1$. Since we also have $f^2_{n_k}$ that converges uniformly on every compact towards f^2, $\exists k_0$ such as : $\forall k \geq k_0$, $\|f^2_{n_k} - f^2\|_{L^\infty(\overline{B(0,\frac{1+r}{2})})} \leq \varepsilon$.
Thus, $\||f_{n_k}|^2 - |f|^2\|_{L^\infty(\overline{B(0,\frac{1+r}{2})})} \leq \varepsilon \Rightarrow |f|^2 \leq |f_{n_k}|^2 + \varepsilon$ on $B(0, \frac{1+r}{2}$.
We obtain : $\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it}|^2 dt \leq \frac{1}{2\pi}\int_0^{2\pi} |f_{n_k}(re^{it}|^2 dt + \frac{1}{2\pi}\int_0^{2\pi} \varepsilon dt \leq 1 + \varepsilon$
$\Rightarrow \sup 0 < s < 1(\frac{1}{2\pi}\int_0^{2\pi} |f(se^{it}|^2 dt) \leq 1 + \varepsilon$.
So f is in H^2, and $\|f\|_{H^2} \leq 1$.
Thus, $(f_{n_k} - f)_k$ is bounded by 2 in H^2-norm and converges towards 0 uniformly on every compact.
So $\|C_\varphi(f_{n_k} - f)\|_{H^2} \to 0$, which implies that $(C_\varphi(f_{n_k}))_k$ converges towards $C_\varphi(f)$ in H^2-norm by the hypothesis.
Thus, $(C_\varphi(f_n)_n$ is a relatively compact sequence in H^2, so C_φ is compact.
$\Longrightarrow$ Let $(f_n)_n \in H^2(\mathbb{D})$ with $\|f_n\|_{H^2} \leq C$, for a $C > 0$, and $f_n \to 0$ uniformly on every compact.
Suppose that $\exists\varepsilon > 0$ such as the set {n with $\|C_\varphi(f_n)\|_{H^2} \geq \varepsilon$} is infinite.
We have then a subsequence $(f_{n_k})_k$ with $\|C_\varphi(f_{n_k})\|_{H^2} \geq \varepsilon$, $\forall k \geq 0$.
Since C_φ is compact, we have $(C_\varphi(f_{n_{k_m}})_m$ that converges towards g in H^2-norm.
Thus, $\|g\|_{H^2} \geq \varepsilon$. But as convergence in H^2-norm implies uniform convergence on every compact, we must have $g \equiv 0$, contradiction.
Thus, $\forall\varepsilon > 0$, $Card(\{$n with $\|C_\varphi(f_n)\|_{H^2} \geq \varepsilon\}) < \infty$.
So, $\forall\varepsilon > 0$, $\exists n_0 > 0$ such as : $\forall n \geq n_0$, $\|C_\varphi(f_n)\|_{H^2} < \varepsilon$, which implies that $C_\varphi(f_n)$ converges towards 0 in H^2-norm.
□

8.7 Note. *This theorem can also be extended to any open simply connected space Ω.*

8.3 Relationship between compactness of C_φ and shape of $Im(\varphi)$

8.8 Proposition. — *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic, with its continuous extension to $\overline{\mathbb{D}}$.*
If $\mu(\{\theta \in [0, 2\pi[$ such as $\varphi(e^{i\theta}) \in \partial\mathbb{D}\}) > 0$, then C_φ isn't compact in $H^2(\mathbb{D})$.

Proof. $(z^n)_n$ is a sequence of functions in H^2 of norm 1 that uniformly converges on every compact towards 0.
We have : $\frac{1}{2\pi}\int_0^{2\pi} |p(e^{it})|^{2n} dt \geq \mu(\{\theta \in [0, 2\pi[$ such as $\varphi(e^{i\theta}) \in \partial\mathbb{D}\}) := c$.
Thus, $C_\varphi(z^n)$ doesn't converge in H^2-norm towards 0, and theorem 8.6 tells us that C_φ isn't compact. □

8.9 Proposition. — *Let Ω be an open simply connected space and $\varphi : \Omega \to \Omega$ holomorphic.*
If $d(Im(\varphi), \Omega^C) > 0$ and $\begin{array}{ll} |\partial\Omega| < \infty & \textit{then } C_\varphi \in s(H^2(\Omega)). \\ |\Omega| < \infty & \textit{then } C_\varphi \in s(B^2(\Omega)). \end{array}$

Proof. The proof will focus on the Hardy space case. The proof for the Bergman space case is identical.
We have : $\int_{\partial\Omega} K_{H^2(\Omega)}(\varphi(z), \varphi(z))|dz| \leq |\partial\Omega|.\|K_{H^2(\Omega)}\|_{L^\infty(\overline{Im(\varphi)} \times \overline{Im(\varphi)})} < \infty$, so C_φ is in $\mathcal{I}_2(H^2(\Omega))$.
Let $\phi : \mathbb{D} \to \Omega$ a biholomorphism. We then have : $d(Im(\phi \circ \varphi \circ \phi^{-1}), \mathbb{D}^C) > 0$. So $\exists r > 0$ such as $Im((1 + r).\phi \circ \varphi \circ \phi^{-1}) \subset \mathbb{D}$.
Let $n > o$. Let $\gamma_n(z) := \frac{1}{1+r}^{\frac{1}{n}}.z$, $\Gamma_n := \phi^{-1} \circ \gamma_n \circ \phi$, $\theta(z) := (1 + r).\phi \circ \varphi \circ \phi^{-1}(z)$.
We have $\gamma_n(\mathbb{D}) \subset \mathbb{D}$ so $\Gamma_n(\Omega) \subset \Omega$, and $\theta(\mathbb{D}) \subset \mathbb{D}$. Thus, the composition operators C_{Γ_n} and $C_{\phi^{-1}\circ\theta\circ\phi}$ are well defined and bounded.
Then $\phi \circ \varphi \circ \phi^{-1} = \gamma_n \circ \ldots \circ \gamma_n \circ ((1 + r).\phi \circ \varphi \circ \phi^{-1})$ and $\varphi = \Gamma_n \circ \ldots \circ \Gamma_n \circ (\phi^{-1} \circ \theta \circ \phi)$.
We have $d(Im(\Gamma_n, \Omega^C) > 0$ and $d(Im(\phi^{-1} \circ \theta \circ \phi), \Omega^C) > 0$. So $C_{\Gamma_n}, C_{\phi^{-1}\circ\theta\circ\phi} \in \mathcal{I}_2(H^2(\Omega))$.
Thus, $C_\varphi \in \mathcal{I}_{\frac{2}{n+1}}(H^2(\Omega))\ \forall n > 0 \Rightarrow C_\varphi \in s(H^2(\Omega))$. □

8.10 Note. *In the case of $\mathbb{D}$, we know that $\varphi : \mathbb{D} \to \mathbb{D}$ holomorphic, is extendable continuously to $\overline{\mathbb{D}}$, and if the image of the boundary of $\mathbb{D}$ by φ is included in $\mathbb{D}$, then C_φ is an operator of type s in both Hardy and Bergman spaces.*
We also know that if $\mu(\{\theta \in [0, 2\pi[$ such as $\varphi(e^{i\theta}) \in \partial\mathbb{D}\}) > 0$, then C_φ is not compact in $H^2(\mathbb{D})$.
Thus, the functions φ for which the compactness properties of C_φ are unclear are only those for which $\varphi(\partial\mathbb{D}) \cap \partial\mathbb{D}$ is of measure 0.

8.11 Proposition. — *Let $f, g \in Hol(\mathbb{D})$, with g injective and $f(\mathbb{D}) \subset g(\mathbb{D})$.*
- If g is in $H^2(\mathbb{D})$ or $B^2(\mathbb{D})$, then so does f.
Let's suppose that $f(\mathbb{D}), g(\mathbb{D}) \subset \mathbb{D}$.
- If C_g is in $I_p(H^2(\mathbb{D}))$ or $I_p(B^2(\mathbb{D}))$ for a $p > 0$, then so does C_f.

Proof. Since g is injective, $g|_{\mathbb{D}}^{g(\mathbb{D})}$ is a biholomorphism and possesses an inverse $h : g(\mathbb{D}) \to \mathbb{D}$. Since $f(\mathbb{D}) \subset g(\mathbb{D})$, we have $f = g \circ (h \circ f)$, with $h \circ f : \mathbb{D} \to \mathbb{D}$, holomorphic.
Thus, $C_{h\circ f}$ is a bounded operator in $H^2(\mathbb{D})$ and $B^2(\mathbb{D})$, which allows us to end the proof. □

Thus, for $\Omega \subset \mathbb{D}$ who has a biholomorphism $g : \mathbb{D} \to \Omega$ that defines a composition operator with good compact properties in $H^2(\mathbb{D})$ or $B^2(\mathbb{D})$, for every $f : \mathbb{D} \to \Omega$ holomorphic, C_f shares these properties.
So the shape of the image of f can automatically induce compactness properties for C_f.

8.12 Corollary. *Let $f, g \in Hol(\mathbb{D})$, with g injective and $f(\mathbb{D}) \subset g(\mathbb{D})$.*
- If f isn't in $H^2(\mathbb{D})$ or $B^2(\mathbb{D})$, then so does g.
Let's suppose that $f(\mathbb{D}), g(\mathbb{D}) \subset \mathbb{D}$.
- If C_f isn't in $I_p(H^2(\mathbb{D}))$ or $I_p(B^2(\mathbb{D}))$ for a $p > 0$, then so does C_g.

8.13 Proposition. — *Let $0 < \lambda < 1$. Let $\varphi : z \in \mathbb{D} \mapsto \lambda z + (1-\lambda) \in \mathbb{D}$. Then C_φ isn't compact in $H^2(\mathbb{D})$.*

Proof. $\forall 0 < r < 1$, we define $f_r(z) := \frac{K_{H^2(\mathbb{D})}(r,z)}{sqrtK_{H^2(\mathbb{D})}(r,r)} = \frac{sqrt1-r}{1-rz}$.
We have $\|f_r\|_{H^2} = 1$, and $f_r \to_{r\to 1^-} 0$ uniformly on every compact.
And $\|f_r \circ \varphi\|_{H^2} = \frac{1+r}{1-r+2\lambda r} \to_{r\to 1^-} \frac{1}{\lambda}$. Thus, $C_\varphi(f_r)$ doesn't converge to 0 in H^2-norm for $r \to 1^-$. Theorem 8.6 tells us that C_φ isn't compact. □

8.14 Note. *- The shape of $Im(\varphi)$ is a disc of radius λ centered in $(1-\lambda)$. φ is a biholomorphism between $\mathbb{D}$ and that disc, and $\varphi(\partial\mathbb{D})$ only touches $\partial\mathbb{D}$ for z=1.*
Thus, we have functions who touch the boundary of the disc in an unique point but who give a non-compact composition operator.
- By composing these φ with rotations, we can obtain every disc included in $\mathbb{D}$ that has one tangential point with $\partial\mathbb{D}$.
Corollary 8.12 tells us that for any $f : \mathbb{D} \to \mathbb{D}$ holomorphic injective with $Im(f)$ that contains a disc tangent to $\partial\mathbb{D}$, C_f isn't compact in $H^2(\mathbb{D})$.

8.15 Proposition. — *For $f(z) = \frac{1-z}{2}$, $Im(z) = B(\frac{1}{2}, \frac{1}{2})$, so remark 8.14 tells us that C_f isn't compact in $H^2(\mathbb{D})$.*
However, $f \circ f(z) = \frac{1+z}{4}$, so $Im(f \circ f) = B(\frac{1}{4}, \frac{1}{4})$ and proposition 8.5 tells us that $C_f \circ C_f = C_{f\circ f}$ is of type s.
Thus, we have a non-compact composition operator whose square possesses great compact properties.

8.16 Proposition. — *- $\forall 0 < \alpha < \frac{1}{2}$, $f_\alpha : z \mapsto (\frac{1+z}{1-z})^\alpha$ is injective and in $H^2(\mathbb{D})$.*
The map $z \mapsto \frac{1+z}{1-z}$ is a biholomorphism from $\mathbb{D}$ to the half-plane $\{z \text{ such as } Re(z) > 0\}$. Thus, the image of $\mathbb{D}$ by f_α is $\{z \text{ such as } |arg(z)| < \pi\alpha\}$, an angular sector with a vertex angle of $\pi\alpha < \frac{\pi}{2}$.
Thus, by composing f_α with roations and translations, and by using property 8.11, we obtain :
For any g in $Hol(\mathbb{D})$ with $Im(g)$ included in an angular sector with vertex angle lower than $\frac{\pi}{2}$, $g \in H^2(\mathbb{D})$.
- For $\alpha = \frac{1}{2}$, $f_\alpha \notin H^2(\mathbb{D})$.
Thus, for any Ω open and simply connected, that contains an angular sector with vertex angle of $\frac{\pi}{2}$, and $g : \mathbb{D} \to \Omega$ a biholomorphism, $g \notin H^2(\mathbb{D})$.
- For $0 < \alpha < 1$, $f_\alpha \in B^2(\mathbb{D})$, and for $\alpha = 1$, $f_\alpha \notin B^2(\mathbb{D})$.
Thus, we have similar properties for $B^2(\mathbb{D})$ but with angular sectors of vertex angler either higher or lower than π.

8.4 Lens maps and regular polygons in $\mathbb{D}$

8.17 Definition. Let $0 < \alpha < 1$. Let $\sigma : z \in \mathbb{D} \mapsto \frac{1+z}{1-z} \in \{Re(z) > 0\}$ a biholomorphism.
We define $\psi_\alpha(z) := \sigma^{-1}(\sigma(z)^\alpha) = \frac{\sigma(z)^\alpha - 1}{\sigma(z)^\alpha + 1}$.
The image of $\mathbb{D}$ by $z \mapsto \sigma(z)^\alpha$ is $\{z \text{ such as } |arg(z)| < \pi\alpha\}$. Thus, the image of ψ_α is a lens-shaped region of $\mathbb{D}$, symmetric to the horizontal axis, that touches $\partial\mathbb{D}$ in 1 and -1. ψ_α is called a **lens map**.
We denote by L_α the image of ψ_α.

8.18 Proposition. — $\forall 0 < \alpha < 1$, $C_{\psi_\alpha} \in \mathcal{I}_2(H^2(\mathbb{D}))$.

Proof. The main issues for the integrability of $\frac{1}{1-|\psi_\alpha(e^{it})|^2}$ are in ± 1. By symmetry, we will only look near $+1$.
We have $\sigma(e^{it}) = \frac{1+e^{it}}{1-e^{it}} = \frac{e^{-i\frac{t}{2}} + e^{i\frac{t}{2}}}{e^{-i\frac{t}{2}} - e^{i\frac{t}{2}}} = i.cotan(\frac{t}{2})$, and $\psi_\alpha(z) = 1 - \frac{2}{\sigma(z)^\alpha + 1}$.
For $|t| < \frac{\pi}{2}$, $|\sigma(e^{it})| = |cotan(\frac{t}{2}| = \frac{1}{|\tan(\frac{t}{2})|} \le \frac{2}{|t|}$.
$\Rightarrow |1 - \psi_\alpha(e^{it})| \ge \frac{2}{|\sigma(e^{it})|^\alpha + 1} \ge C.|t|^\alpha$.
Since $\psi_\alpha(e^{it})$ approaches $+1$ non-tangentially to $\partial\mathbb{D}$, we can also obtain that for t near 0, $1 -$

$|\psi_\alpha(e^{it})|^2 \geq \widetilde{C}.|1-\psi_\alpha(e^{it})|$, for a $\widetilde{C} > 0$.
Thus, $\frac{1}{1-|\psi_\alpha(e^{it})|^2} \geq \frac{1}{C.\widetilde{C}} \cdot \frac{1}{|t|^\alpha}$ for t near 0, so this function is integrable near 0.
Therefore, $\frac{1}{1-|\psi_\alpha(e^{it})|^2}$ on $\partial\mathbb{D}$ and the Hilbert-Schmidt critera in H^2 concludes the proof. $\square$

8.19 Corollary. *Let $f : \mathbb{D} \to \mathbb{D}$ with $Im(f) \subset L_\alpha$ for a $0 < \alpha < 1$.*
Then, $C_f \in \mathcal{I}_2(H^2(\mathbb{D}))$.

8.20 Theorem. *Let Γ be a polygon inscribed in $\mathbb{D}$. Let $f : \mathbb{D} \to \mathbb{D}$ with $Im(f) \subset \Gamma$.*
Then, $C_f \in \mathcal{I}_2(H^2(\mathbb{D}))$.

Proof. Let Γ be a polygon inscribed in $\mathbb{D}$. We will suppose that one of the vertexes of Γ is $+1$.
Let $\phi : \mathbb{D} \to \mathring{\Gamma}$ a biholomorphism. ϕ extends to $\partial\mathbb{D}$ from $\partial\Gamma$ as an homeomorphism.
By composing ϕ with a biholomorphism of the disc, we can send $\phi(1)$ into 1. Thus, we will suppose at $\phi(1) = 1$.
The map $\psi := \frac{1+\phi}{2}$ will then fix 1 and send $\mathbb{D}$ into a lens L_α, α near to 1.
Thus, by corollary 8.19, $C_\psi \in \mathcal{I}_2(H^2(\mathbb{D}))$.
When t is near 0, $\phi(e^{it})$ and $\psi(e^{it})$ approach $+1$ non-tangentially to $\partial\mathbb{D}$.
Thus, $1-|\psi(e^{it})|^2\ |1-\psi(e^{it})| = |\frac{1-\phi(e^{it})}{2}|\ \frac{1-|\phi(e^{it})|^2}{2}$ for t near 0.
Since C_ψ is in $\mathcal{I}_2$, $\frac{1}{1-|\psi(e^{it})|^2}$ is integrable for t near 0, so $\frac{1}{1-|\phi(e^{it})|^2}$ is integrable for t near 0 too.
Thus, $\frac{1}{1-|\phi(e^{it})|^2}$ is integrable over small intervals centered around the preimage of each vertex of Γ, who are exactly all the points where $|\phi(e^{it})| = 1$. This means that this function is integrable over $[0, 2\pi[$.
Therefore, $C_\phi \in \mathcal{I}_2(H^2(\mathbb{D}))$.
We then use proposition 8.11 to conclude the proof.

$\square$

9 Composition operators on $H^p(\Omega), B^p(\Omega)$

9.1 Existence of compact composition operators on $H^p(\Omega), B^p(\Omega)$

9.1 Definition. Let $0 < p < \infty$. Let Ω be an open and simply connected space, and $\psi : \mathbb{D} \to \Omega$ a biholomorphism. We define : $B^p(\Omega) := \{f \in Hol(\Omega)$ such as $\iint_\Omega |f(z)|^p|dz| < \infty\}$ $H^p(\Omega) := \{f \in Hol(\Omega)$ such as $\sup_{r\to 1^-}(\int_{\psi(\partial B(0,r))} |f(z)|^p|dz| < \infty\}$.

$H^p(\Omega)$ and $B^p(\Omega)$ are Banach spaces. They are Hilbert spaces for $p = 2$.
For any $q > 0$, $\psi'^q : \mathbb{D} \to \mathbb{C}$ can be well defined.
Thus, $U_p : f \in B^p(\Omega) \mapsto (f\circ\psi).(\psi')^{\frac{2}{p}} \in B^p(\mathbb{D})$ and $V_p : f \in H^p(\Omega) \mapsto (f\circ\psi).(\psi')^{\frac{1}{p}} \in H^p(\mathbb{D})$ are unitary maps.

For $\varphi : \Omega \to \Omega$ holomorphic and $\phi := \psi^{-1}\circ\varphi\circ\psi$, we define : $A_{B^p,\varphi} := U_p C_\varphi U_p^{-1} = M_{(\frac{\psi'}{\psi'\circ\phi})^{\frac{2}{p}}} C_\phi$ and $A_{H^p,\varphi} := V_p C_\varphi V_p^{-1} = M_{(\frac{\psi'}{\psi'\circ\phi})^{\frac{1}{p}}} C_\phi$.

, we can then study properties of C_φ on $H^p(\Omega)$ (resp $B^p(\Omega)$) by studying $A_{H^p,\varphi}$ on $H^p(\mathbb{D})$ (resp $A_{B^p,\varphi}$ on $B^p(\mathbb{D})$).

We can now introduce the main theorem of this section, and give the key points of its proof.

9.2 Theorem. *Existence of compact composition operators in $B^p(\Omega), H^p(\Omega)$*
There exist compact composition operators in $B^p(\Omega) \Leftrightarrow |\partial\Omega|$ is finite.
There exist compact composition operators in $H^p(\Omega) \Leftrightarrow |\Omega|$ is finite.

For $\varphi : \Omega \to \Omega$ holomorphic, instead of focusing on C_φ, we will be focusing on $A_{B^p,\varphi}$ and $A_{H^p,\varphi}$, in order to go back to $\mathbb{D}$.

9.3 Note. $|\Omega| < \infty \Leftrightarrow \iint_{\mathbb{D}} |\psi'(z)||dz| < \infty \Leftrightarrow \psi' \in B^1(\mathbb{D})$
$|\partial\Omega| < \infty \Leftrightarrow \sup_{r\to 1^-}(\int_{\partial B(0,r)} |\psi'(z)||dz|) < \infty \Leftrightarrow \psi' \in B^1(\mathbb{D})$
Thus, we can trade the finiteness of the area (resp border) of Ω by ψ' belonging to a certain Bergman (resp Hardy) space.
This condition does also not depend on the biholomorphism ψ chosen.

The next propositions and theorems will be similar for $H^p(\Omega)$ and $B^p(\Omega)$. Thus, they will only be stated for $H^p(\Omega)$ to reduce redundancy.

9.4 Theorem. *Let $\varphi : \Omega \to \Omega$, holomorphic.*
If C_φ is bounded in $H^p(\Omega)$ for a $0 < p < \infty$, it is bounded in $H^p(\Omega)$ for all $0 < p < \infty$.

9.5 Note.
- The proof of this theorem focuses on $A_{H^p,\varphi}$ and shows that for any $0 < p, q < \infty$, $\|A_{H^p,\varphi}\|^p_{H^p(\mathbb{D})} = \|A_{H^p,\varphi}\|^q_{H^q(\mathbb{D})}$.

9.6 Lemma. *Let $\varphi : \Omega \to \Omega$, holomorphic.*
- $A_{H^p,\varphi}$ is compact in $H^p(\mathbb{D}) \iff \forall (f_n)_n \in H^p(\mathbb{D})^{\mathbb{N}}$ such as :
$\|f_n\|_{H^p}$ is uniformly bounded by a constant $C > 0$
$f_n \to 0$ uniformly on every compact
, we have $\|A_{H^p,\varphi}(f_n)\|_{H^p} \to 0$.
- If $A_{H^p,\varphi}$ is compact, then $\mu(\partial\mathbb{D} \cap \varphi(\mathbb{D})) = 0$.
We use the family $\{z^n\}_n$ and the first part of the lemma to prove this part.
- C_φ is compact in $H^p(\Omega)$ for a $0 < p < \infty \Leftrightarrow$ it is compact in $H^p(\Omega)$ for all $0 < p < \infty$.

9.7 Note.
- We can now focus on $H^2(\Omega)$ and use its inner product and RKHS properties to study boundedness and compactness of composition operators.
- We can prove the left part of the main theorem :
$\Longleftarrow$ If $|\Omega|$ is finite, constant functions are in $H^2(\Omega)$. Thus, $\forall w \in \Omega$, $C_w : f \in H^2(\Omega) \mapsto f(w) \in H^2(\Omega)$ is bounded. C_w being a finite rank operator, it is compact.
- We are now left with the right part of the theorem to prove, in the case p=2.

9.8 Theorem. *Let Ω open and simply connected, and $\psi : \mathbb{D} \to \Omega$ a biholomorphism.*
If $H^2(\Omega)$ contains a compact composition operator, then $\psi' \in H^1(\mathbb{D})$.

9.9 Lemma. *Let $\varphi : \Omega \to \Omega$, holomorphic, and $\phi := \psi^{-1} \circ \varphi \circ \psi$.*
Let $K^{H^2(\mathbb{D})}$ be the reproducing kernel of $H^2(\mathbb{D})$. $\forall z \in \mathbb{D}$, we have $k_z^{H^2(\mathbb{D})}(.) = \overline{K^{H^2(\mathbb{D})}(z,.)} \in H^2(\mathbb{D})$.
If $A_{H^2,\varphi}$ is bounded in $H^2(\mathbb{D})$, then $A^*_{H^2,\varphi}(k_z^{H^2(\mathbb{D})}) = \overline{(\frac{\psi'}{\psi'\circ\phi})^{\frac{1}{2}}}(z).k_{\phi(z)}^{H^2(\mathbb{D})}$.

Proof. For any $f \in H^2(\mathbb{D})$, we have : $\langle A^*_{H^2,\varphi}(k_z^{H^2(\mathbb{D})}), f\rangle = \langle k_z^{H^2(\mathbb{D})}, A_{H^2,\varphi}(f)\rangle = A_{H^2,\varphi}(f)(z)$.
And $A_{H^2,\varphi}(f)(z) = \overline{(\frac{\psi'}{\psi'\circ\phi})^{\frac{1}{2}}}(z).f(\phi(z)) = \langle \overline{(\frac{\psi'}{\psi'\circ\phi})^{\frac{1}{2}}}(z).k_{\varphi(z)}^{H^2(\mathbb{D})}, f\rangle$, by using the properties of $k_w^{H^2(\mathbb{D})}$. □

We will now prove theorem 9.8 when $\varphi : \Omega \to \Omega$ has a fix point. ($\phi = \psi^{-1}\circ\varphi\circ\psi$ has a fix point too)

Proof. Theorem 9.8
Let $\varphi : \Omega \to \Omega$ holomorphic such as C_φ is compact in $H^2(\Omega)$.
Suppose that we have $a \in \Omega$ such as $\varphi(a) = a$.
For $b := \psi^{-1}(a)$, we have $\phi(b) = b$.
As C_φ is compact, φ can't be bijective, so ϕ is also not bijective. Lemma 9.9 gives us : $A^*_{H^2,\varphi}(k_b^{H^2(\mathbb{D})}) =$

$\overline{(\frac{\psi'}{\psi'\circ\phi})^{\frac{1}{2}}}(b).k_{\phi(b)}^{H^2(\mathbb{D})} = 1.k_b^{H^2(\mathbb{D})}$. Thus, 1 is an eigenvalue of $A^*_{H^2,\varphi} \Rightarrow 1 \in \sigma(A_{H^2,\varphi})$.
Since C_φ is compact in $H^2(\Omega)$, $A_{H^2,\varphi}$ is compact in $H^2(\mathbb{D})$. So $\exists f \in H^2(\mathbb{D})$ such as $A_{H^2,\varphi}(f) = f$.
For $g := \frac{f}{(\psi')^{\frac{1}{2}}}$, we have $g \circ \phi = g$.
Since $\phi : \mathbb{D} \to \mathbb{D}$ is holomorphic, non- bijective, and fixes b, corollary 5.8 tells us that $\phi_n := \phi \circ \ldots \phi$ converges uniformly on every compact towards the constant function b.
So $g(z) = g(\phi_n(z))$, $\forall n > 0$, $\forall z \in \mathbb{D} \Rightarrow g(z) = g(b)$, $\forall z \in \mathbb{D} \Rightarrow g \equiv g(b)$.
We end up with $f = g(b).(\psi')^{\frac{1}{2}} \in H^2(\mathbb{D})$. As $f \neq 0$, we have $g(b) \neq 0$ and $(\psi')^{\frac{1}{2}} \in H^2(\mathbb{D})$.
Thus, $\psi' \in H^1(\mathbb{D})$. □

The following theorem will complete the proof of theorem 9.8. This will also complete the proof of the main result of this section.

9.10 Theorem. *Let Ω be an open and simply connected space. Let $\varphi : \Omega \to \Omega$ holomorphic.*
If C_φ is compact in $H^2(\Omega)$, then φ has a fix point in Ω.

However, proving theorem 9.10 is the most difficult point of the section[2].

9.2 Other properties for composition operators

We can now use the main theorem of section 9.1, as well as some sub-theorems, in order to obtain interesting results for elements seen in previous parts.

9.11 Proposition. —
$\forall \varphi : \{Re(z) > 0\} \to \{Re(z) > 0\}$ holomorphic, we have $\int_0^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{4\pi Re(\varphi(x+iy))^2} dxdy = +\infty$ and $\sup_{r\to 0^+}(\int_{-\infty}^{+\infty} \frac{1}{2Re(\varphi(r+iy))} dy) = +\infty$.
$\forall \varphi : \{|Im(z)| < \frac{\pi}{2}\} \to \{|Im(z)| < \frac{\pi}{2}\}$ holomorphic, we have $\int_{-\infty}^{+\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4\pi\cos(Im(\varphi(x+iy)))^2} dxdy = +\infty$ and $\sup_{r\to\frac{\pi}{2}^-}(\int_{-\infty}^{+\infty} \frac{1}{2\cos(Im(\varphi(x+ir)))} + \frac{1}{2\cos(Im(\varphi(x-ir)))} dx) = +\infty$.

Proof. Since Ω_1 and Ω_2 have an infinite area and an infinite border, their corresponding Hardy and Bergman spaces hold no compact composition operators.
Thus, they hold no $\mathcal{I}_2$ composition operators neither.
Thus, the Hilbert-Schmidt criteria from 8.3 must always fail.
By using the expression of the Reproducing Kernels computed in section 2.2, we obtain the desired result. □

9.12 Theorem. *Let Ω an open and simply connex space, $\psi : \mathbb{D} \to \Omega$ a biholomorphism, $\varphi : \Omega \to \Omega$ holomorphic, and $\phi := \psi^{-1} \circ \varphi \circ \psi$.*
- If ϕ has a fix point α and if $A_{H^2,\varphi}$ is bounded on $H^2(\mathbb{D})$, then $\{\phi'(\alpha)^n, n > 0\} \cup \{1\}$ are in the spectrum $A_{H^2,\varphi}$).
If $A_{H^2,\varphi}$) has an eigenvalue, then it is of the form $\phi'(0)^n$ for a $n \geq 0$, and has a multiplicity of 1. - If $A_{H^2,\varphi}$ is compact on $H^2(\mathbb{D})$, then ϕ has an unique fix point α and $\{\phi'(\alpha)^n, n > 0\} \cup \{0, 1\} = \sigma(A_{H^2,\varphi})$.

Proof. We conjugate ϕ by a biholomorphism of the disc to send α to 0.
We look at $A_{H^2,\varphi}(z^n)(w) = \frac{\psi'}{\psi'\circ\phi})^{\frac{1}{2}}(w).\phi(w)^n$.
As we have $\phi(0) = 0$, the Taylor series of this function in 0 begins with $(\frac{\psi'(0)}{\psi'\circ\phi(0)})^{\frac{1}{2}}.\phi'(0)^n).z^n = \phi'(0)^n.z^n$.
Thus, the matrix of $A_{H^2,\varphi}$ in the orthonormal basis $\{z^n\}$ is lower triangular with $\phi'(0)^n$ in the diagonal.

[2]See [9], Hardy spaces that support no compact composition operators, p.62-89.

Thus, the matrix of $A^*_{H^2,\varphi}$ in the orthonormal basis $\{z^n\}$ is upper triangular with $\overline{\phi'(0)^n}$ in the diagonal.
So $\overline{\phi'(0)^n}$ is an eigenvalue for $A^*_{H^2,\varphi}$, $\forall n > 0$. 1 is also an eigenvalue of $A^*_{H^2,\varphi}$ with $k_0^{H^2(\mathbb{D})}$ as an eigenvector.
So $\phi'(0)^n$ is in $\sigma(A_{H^2,\varphi})$, $\forall n > 0$, as well as 1.
Lastly, if $A_{H^2,\varphi}(f) = \lambda.f$, then for $g = \frac{f}{(\psi')^{\frac{1}{2}}}$ we have g holomorphic and $g \circ \phi = \lambda.g$.
Koenigs Theorem in 5.9 tells us that $\lambda = \phi'(0)^n$ for a $n \geq 0$ and that this eigenvalue is of multiplicity one. □

9.13 Theorem.
Let Ω an open and simply connex space, $\varphi : \Omega \to \Omega$ holomorphic, with C_φ compact on $H^2(\Omega)$.
1) φ has an unique fix point $\alpha \in \Omega$, and $\sigma(C_\varphi) = \{\varphi'(\alpha)^n,\ n > 0\} \cup \{0, 1\}$.
$\forall n \geq 0$, $\varphi'(\alpha)^n$ is an eigenvalue of multiplicity 1. For σ an eigenvector of $\varphi'(\alpha)$, σ^n is an eigenvector of $\varphi'(\alpha)^n$.
2) There exists c>0 such as for $\alpha_r(C_\varphi)$ the r-th approximation number of C_φ, we have : $\alpha_r(C_\varphi) \geq e^{-c.r}$.

Proof. 1) As C_φ compact on $H^2(\Omega)$, $A_{H^2,\varphi}$ is compact on $H^2(\mathbb{D})$. We can then apply theorem 9.12 to get all the desired properties on C_p as it is a conjugated of $A_{H^2,\varphi}$ by an unitary map.
2) Since C_φ is compact, φ can't be bijective. Thus, the generalized Koenigs theorem tells us that $|\varphi'(\alpha)| < 1$.
This means that the sequence of $|\varphi'(\alpha)|^n$ is monotone decreasing towards 0.
As the multiplicity of these eigenvalues is 1, their associated eigenspace is of dimension 1.
By looking at the subspace engendered by the sum of the eigenspaces and at its orthogonal, we can show that $\forall r \geq 0$, $\alpha_r(C_\varphi) \geq |\varphi'(\alpha)|^r = e^{-ln(\frac{1}{|\varphi'(\alpha)|}.r}$. □

9.14 Note. *We have a good amount of information on the eigenfunctions of φ with Koenigs theorem.*
For C_φ compact and σ an eigenfunction, σ^n must be in $H^2(\Omega)$.
Thus, if σ grows to ∞ near some points of Ω, then its growth rate near such a point must be lower than the growth rate of every $\frac{1}{z^{\frac{1}{n}}}$ near 0.

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